Settling velocities definition for global mass conservation of polydisperse sedimentation models

E.D. Fernández-Nieto

Universidad de Sevilla
http://personal.us.es/edofer

A joint work with R. Bürger and V. Osores (U. Concepción, Chile)

Balance laws in fluid mechanics, geophysics, biology
(theory, computation and application)
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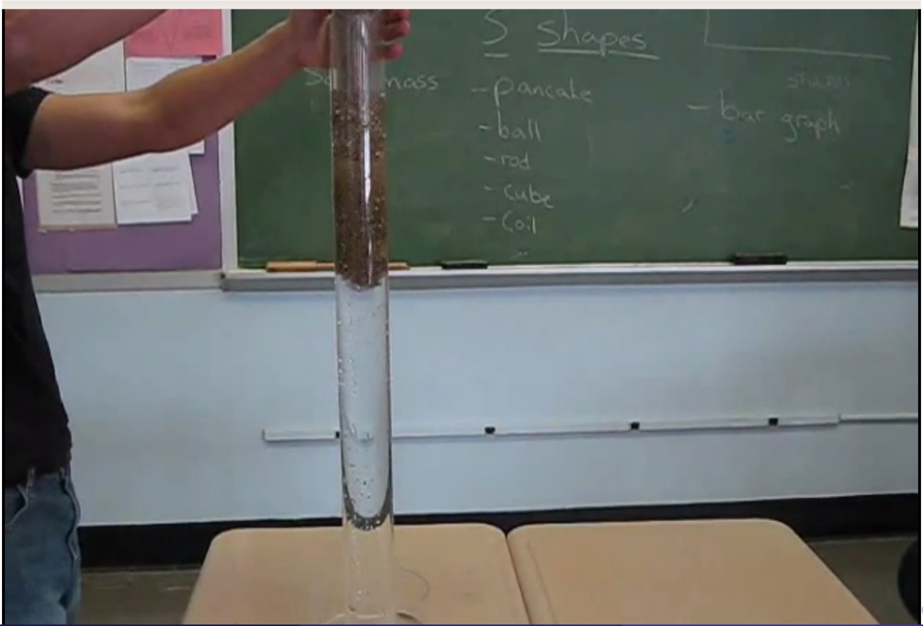
Polydisperse sedimentation



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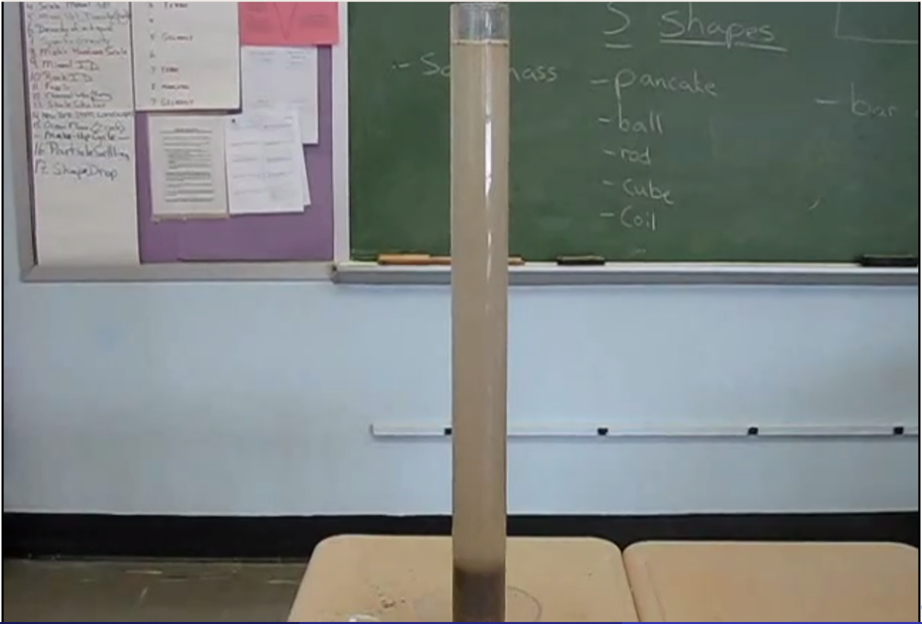

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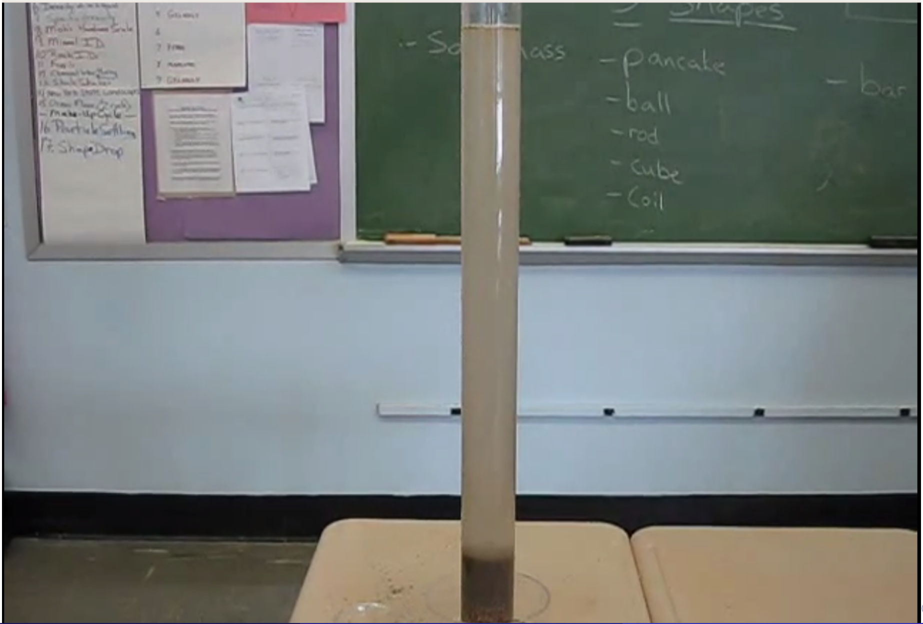

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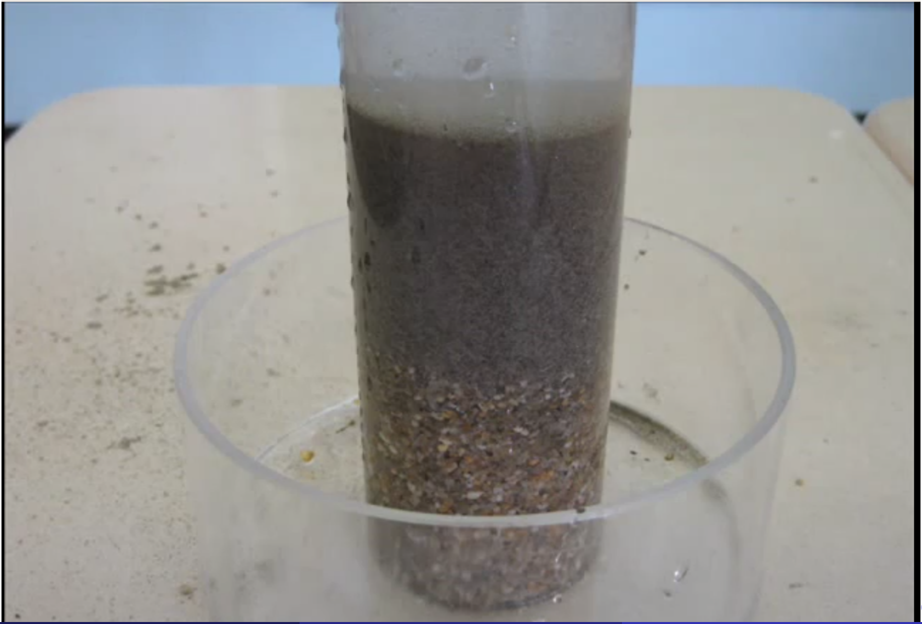

Some related references

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- R. Bürger, R. Donat, P. Mulet, and C. A. Vega, *Hyperbolicity analysis of polydisperse sedimentation models via a secular equation for the flux Jacobian*, SIAM J. Appl. Math., 70 (2010), pp. 2186–2213.

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N sediment species with density ρ_j and size d_j
ϕ_j : volumetric concentration $j = 1, \ldots, N$

$$\phi = \sum_{j=1}^{N} \phi_j, \quad \phi_0 = 1 - \phi, \text{ and } \Phi = (\phi_0, \phi_1, \ldots, \phi_N)$$

$$\rho(\Phi) = \sum_{j=0}^{N} \rho_j \phi_j$$

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$$\mathbf{v}_j = (u_j, w_j), \quad j = 0, 1, \ldots, N,$$

Averaged velocity

$$\bar{\mathbf{v}} = \sum_{j=0}^{N} \phi_j \mathbf{v}_j$$

Relative/slip velocity

$$\Delta \mathbf{v}_j = \mathbf{v}_j - \mathbf{v}_0, \quad j = 1, \ldots, N$$

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Local mass balance equations

Global mass conservation:

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0.$$

The N solid species and the fluid satisfy,

$$\partial_t(\rho_j \phi_j) + \nabla \cdot (\rho_j \phi_j \boldsymbol{v}_j) = 0$$

Then,

$$\partial_t \phi_j + \nabla \cdot (\phi_j \boldsymbol{v}_j) = 0, \quad j = 0, 1, \ldots, N,$$

Sum of all equations:

$$\nabla \cdot \bar{\boldsymbol{v}} = 0 \qquad \left(\bar{\boldsymbol{v}} = \sum_{j=0}^{N} \phi_j \boldsymbol{v}_j \right)$$

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MLB model

$$\partial_t \phi_j + \nabla \cdot (\phi_j \boldsymbol{v}_j) = 0, \quad i = 0, 1, \ldots, M,$$

Sherman-Morrison formula

$$\Delta \boldsymbol{v}_j = \frac{\phi}{\alpha_j(\Phi)} \left[(\rho_j - \rho(\Phi)) g\vec{k} + \frac{\sigma_e(\phi)}{\phi_j} \nabla \left(\frac{\phi_j}{\phi} \right) + \frac{1-\phi}{\phi} \nabla \sigma_e(\phi) \right]$$

$\sigma_e(\phi) = 0$ if $\phi < \phi_c$ (effective solid stress).

MLB model

$$\partial_t \phi_j + \nabla \cdot \left(\phi_j \Delta \boldsymbol{v}_j + \phi_j \bar{\boldsymbol{v}} - \phi_j \sum_{k=1}^{N} \phi_k \Delta \boldsymbol{v}_k \right) = 0, \quad j = 0, 1, \dots, N,$$

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$\sigma_e(\phi) = 0$ if $\phi < \phi_c$ (effective solid stress).

MLB model

Masliyah-Lockett-Bassoon

$$\frac{\phi}{\alpha_j(\Phi)} = -d_j^2 \frac{V(\Phi)}{18\mu_f}$$

μ_f viscosity of pure fluid

$V(\Phi) = (1-\phi)^{n-2}$, $(n > 2)$ hindered settling factor

Masliyah-Lockett-Bassoon

If we consider the case $\phi < \phi_c$ then $\sigma_e = 0$. Then it can be written as follows:

$$\Delta v_j = \mu \delta_j V(\phi)(\overline{\rho}_j - \sum_{k=1}^{N} \overline{\rho}_k \phi_k)\vec{k}$$

Where:

- $V(\phi) = (1-\phi)^{n-2}, n > 2.$
- $\overline{\rho_j} = \rho_j - \rho_0$
- $\mu = -g\dfrac{d_1^2}{18\mu_f}$
- $\delta_j = \dfrac{d_j^2}{d_1^2}$

for $j = 1, \ldots, N,$

Finally, as

$$\phi_j \boldsymbol{v}_j = \phi_j \Delta \boldsymbol{v}_j + \phi_j \bar{\boldsymbol{v}} - \phi_j \sum_{k=1}^{N} \phi_k \Delta \boldsymbol{v}_k$$

we can write

$$\phi_j \boldsymbol{v}_j = f_j(\Phi)\vec{k} + \phi_j \bar{\boldsymbol{v}}$$

where

$$f_j(\phi) = \mu V(\phi)\phi_j \left(\delta_j(\overline{\rho}_j - \sum_{k=1}^{N} \overline{\rho}_j \phi_j) - \sum_{l=1}^{N} \delta_l \phi_l \left(\overline{\rho}_l - \sum_{k=1}^{N} \overline{\rho}_k \phi_k \right) \right).$$

$$\partial_t \phi_j + \partial_z (f_j(\phi)) = 0, \quad j = 1, \dots, N$$

$$f_j(\phi) = \phi_j \mu V(\phi) \left(\delta_j (\bar{\rho}_j - \bar{\boldsymbol{\rho}}^{\mathrm{T}} \Phi) - \sum_{k=1}^{N} \phi_k \delta_k (\bar{\rho}_k - \bar{\boldsymbol{\rho}}^{\mathrm{T}} \Phi) \right), \quad j = 1, \dots, N.$$

Global mass conservation of MLB model

Continuity equation for each specie $\left(\bar{\boldsymbol{v}} = (\bar{u}, \bar{w}) \right)$:

$$\partial_t \phi_j + \partial_x(\phi_j \bar{u}) + \partial_z(\phi_i \bar{w} + f_i(\phi)) = 0, \qquad j = 1, \ldots, N$$

with

$$\mathrm{div} \bar{v} = 0$$

Mass conservation

We cannot conclude from this definition of MLB model the global mass conservation

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0.$$

Because of the definition of the averaged velocity

$$\bar{\boldsymbol{v}} = \sum_{j=0}^{N} \phi_j \boldsymbol{v}_j.$$

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Mass average velocity

We consider the mass average velocity of the mixture

$$\bar{\boldsymbol{v}} := \frac{1}{\rho}\sum_{j=0}^{N}\rho_j\phi_j\boldsymbol{v}_j = \frac{1}{\rho}\left[\left(\rho - \sum_{j=1}^{N}\rho_j\phi_j\right)\boldsymbol{v}_0 + \sum_{k=1}^{N}\rho_k\phi_k\boldsymbol{v}_k\right],$$

which satisfies the global mass balance of the mixture

$$\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0. \tag{1}$$

Modified MLB model

Defining the slip velocities

$$\Delta \boldsymbol{v}_j := \boldsymbol{v}_j - \boldsymbol{v}_0$$

and

$$\lambda_j := \frac{\rho_j \phi_j}{\rho} \qquad \text{for} \quad j = 1, \dots, N,$$

we derive the identity

$$\phi_j \boldsymbol{v}_j = \phi_j \big(\Delta \boldsymbol{v}_j + \bar{\boldsymbol{v}} - (\lambda_1 \Delta \boldsymbol{v}_1 + \cdots + \lambda_N \Delta \boldsymbol{v}_N) \big), \quad j = 1, \dots, N; \tag{2}$$

Modified MLB

By following the steps that in the deduction of MLB model we get

$$\phi_j \boldsymbol{v}_j = f_j^{\mathrm{M}}(\Phi)\boldsymbol{k} + \phi_j \bar{\boldsymbol{v}} \quad \text{for } j = 1, \dots, N,$$

where

$$f_j^{\mathrm{M}}(\Phi) := \phi_j v_j^{\mathrm{MLB}} = \phi_j \mu V(\phi) \left(\delta_j (\bar{\rho}_j - \bar{\boldsymbol{\rho}}^{\mathrm{T}} \Phi) - \sum_{k=1}^{N} \lambda_k \delta_k (\bar{\rho}_k - \bar{\boldsymbol{\rho}}^{\mathrm{T}} \Phi) \right), \quad j = 1, \dots, N. \quad (3)$$

Finally, the continuity equation can be written as

$$\partial_t \phi_j + \nabla \cdot \left(\phi_j \bar{\boldsymbol{v}} + f_j^{\mathrm{M}}(\Phi)\boldsymbol{k} \right) = 0, \quad j = 1, \dots, N.$$

what implies the global mass conservation.

Influence on the vertical velocity of the fluid

Note that the vertical velocities of particles satisfy

$$\rho_j \phi_j w_j = \rho_j \phi_j w + \rho_j f_j^{\mathrm{M}}(\Phi),$$

moreover we have the identity

$$\sum_{j=1}^{N} \lambda_j w_j = (1 - \lambda_0) w + \frac{1}{\rho} \sum_{j=1}^{N} \rho_j f_j^{\mathrm{M}}$$

that can be rearranged as

$$\lambda_0 w_0 = \lambda_0 w - \frac{1}{\rho} \sum_{j=1}^{N} \rho_j f_j^{\mathrm{M}}.$$

That is,

$$\rho_0 \phi_0 w_0 = \rho_0 \phi_0 \bar{\boldsymbol{v}} - \sum_{j=1}^{N} \rho_j f_j^{\mathrm{M}}(\Phi).$$

Final form of the model equations

- With $\phi_j \boldsymbol{v}_j = \phi_j \bar{\boldsymbol{v}} + f_j^{\mathrm{M}}(\Phi)\vec{k}$,

$$\partial_t(\rho_j \phi_j) + \nabla \cdot (\rho_j \phi_j \boldsymbol{v}_j) = 0, \quad j = 1, N,$$
$$\partial_t(\rho_j \phi_j \boldsymbol{v}_j) + \nabla \cdot (\rho_j \phi_j \boldsymbol{v}_j \otimes \boldsymbol{v}_j) = \nabla \cdot \boldsymbol{T}_j - \phi_j \rho g \vec{k}, \quad j = 0, \ldots, N.$$

- Summing up from 0 to N the momentum balance equations,

$$\partial_t \left(\sum_{j=0}^{N} \rho_j \phi_j \right) + \nabla \cdot \left(\sum_{j=0}^{N} \rho_j \phi_j \boldsymbol{v}_j \right) = 0.$$

$$\partial_t \left(\sum_{j=0}^{N} \rho_j \phi_j \boldsymbol{v}_j \right) + \nabla \cdot \left(\sum_{j=0}^{N} \rho_j \phi_j \boldsymbol{v}_j \otimes \boldsymbol{v}_j \right) = \nabla \cdot \boldsymbol{T} - \rho g \vec{k},$$

with $\boldsymbol{T} = \sum_{j=0}^{N} \boldsymbol{T}_j$.

- Then,

$$(*) \ \Rightarrow \ \partial_t \rho + \nabla \cdot (\rho \bar{\boldsymbol{v}}) = 0$$
$$(**) \ \Rightarrow \ \partial_t(\rho \bar{\boldsymbol{v}}) + \nabla \cdot (\rho \bar{\boldsymbol{v}} \otimes \bar{\boldsymbol{v}}) = \nabla \cdot \boldsymbol{\Sigma} - \rho g \vec{k},$$

with $\boldsymbol{\Sigma} := \boldsymbol{T} - \sum_{j=0}^{N} \rho_j \phi_j (\boldsymbol{v}_j - \bar{\boldsymbol{v}}) \otimes (\boldsymbol{v}_j - \bar{\boldsymbol{v}})$.

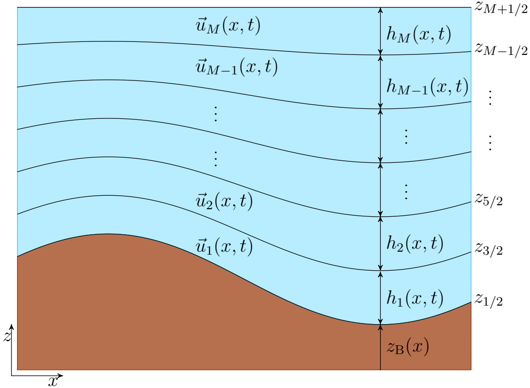

Figure: Model problem.

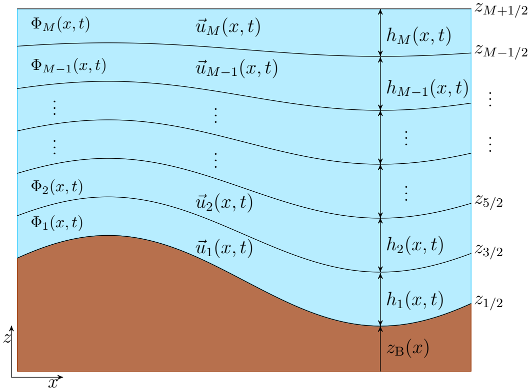

Figure: Model problem.

Definition (Weak solution)

Assume $\vec{v}_1, \ldots, \vec{v}_N$, p, and $\phi_1, \ldots, \phi_N$ are smooth in each $\Omega_\alpha(t)$. Then $\vec{y} := (\vec{v}_1, \ldots, \vec{v}_N, \phi_1, \ldots, \phi_N, p)$ is a *weak solution* if:

(i) $\vec{y}$ is a standard weak soln in each $\Omega_\alpha(t)$,

(ii) normal flux jump conditions across each $\Gamma_{\alpha+1/2}(t)$ are satisfied:

$$\left[\!\left[(\rho_j\phi_j; \rho_j\phi_j\vec{v}_j)\right]\!\right]_{t,\alpha+1/2} \cdot \vec{n}_{t,\alpha+1/2} = 0 \quad \textit{for all } j = 1, \ldots, N,$$

$$\left[\!\!\left[\left(\sum_{l=0}^{N} \rho_l\phi_l\vec{v}_l; \sum_{l=0}^{N} \rho_l\phi_l\vec{v}_l \otimes \vec{v}_l - \boldsymbol{T}\right)\right]\!\!\right]_{t,\alpha+1/2} \cdot \vec{n}_{t,\alpha+1/2} = 0.$$

E. Audusse, M. Bristeau, B. Perthame, J. Sainte-Marie. *A multilayer Saint-Venant system with mass exchanges for shallow water flows. derivation and numerical validation.* ESAIM: Mathematical Modelling and Numerical Analysis, **45** (2011) 169–200.

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Multilayer approach

- Assume that $h_\alpha = l_\alpha h$ for $\alpha = 1, M$, $l_\alpha > 0$, $l_1 + \cdots + l_M = 1$.
- Define for $\alpha = 1, M$

$$r_{j,\alpha} := \rho_j \phi_{j,\alpha} h, \quad j = 0, N; \quad q_\alpha := \bar{\rho}_\alpha h u_\alpha, \quad m_\alpha := \bar{\rho}_\alpha h.$$

Governing model, final form ($\alpha = 1, M, j = 1, N$):

$$\partial_t m_\alpha + \partial_x q_\alpha = (G_{\alpha+1/2} - G_{\alpha-1/2})/l_\alpha, \qquad \Rightarrow \partial_t \bar{m} + \partial_x \Big(\sum_{\alpha=1}^{M} l_\alpha q_\alpha\Big) = 0.$$

$$\partial_t r_{j,\alpha} + \partial_x \left(\frac{r_{j,\alpha} q_\alpha}{m_\alpha}\right) = \frac{1}{l_\alpha}\left(\tilde{\phi}_{j,\alpha+1/2} G_{\alpha+1/2} - \tilde{\phi}_{j,\alpha-1/2} G_{\alpha-1/2}\right) - \frac{\rho_j}{l_\alpha} \Delta_\alpha \tilde{f}_{j,\alpha+1/2},$$

$$\partial_t q_\alpha + \partial_x \left(\frac{q_\alpha^2}{m_\alpha} + h\left(\frac{g}{2} l_\alpha m_\alpha + g \sum_{\beta=\alpha+1}^{M} l_\beta m_\beta\right)\right) = g \sum_{\beta=\alpha+1}^{M} l_\beta m_\beta \partial_x h - g m_\alpha \partial_x z_b$$
$$- g m_\alpha L_{\alpha-1} \partial_x h + \left(\tilde{u}_{\alpha+1/2} G_{\alpha+1/2} - \tilde{u}_{\alpha-1/2} G_{\alpha-1/2}\right)/l_\alpha.$$

Compact form

$$\partial_t \vec{w} + \partial_x \boldsymbol{\mathcal{F}}(\vec{w}) = \boldsymbol{\mathcal{S}}(\vec{w}, \partial_x(\vec{w})) + \boldsymbol{\mathcal{G}}(\vec{w}, \partial_x(\vec{w})), \tag{4}$$

$$\vec{w} = (\bar{m}, \{q_\alpha\}_{\alpha=1}^{M}, r_{11}, \ldots, r_{N1}, \ldots, r_{1,\alpha}, \ldots, r_{N,\alpha}, r_{1,M}, \ldots, r_{N,M}).$$

- $G_{j,\alpha+1/2} = \tilde{\phi}_{j,\alpha+1/2} G_{\alpha+1/2} - \rho_j \tilde{f}_{j,\alpha+1/2}$,

$$\tilde{\phi}_{j,\alpha+1/2} = \frac{1}{2}\left(\frac{\rho_j \phi_{j,\alpha+1}}{\bar{\rho}_{\alpha+1}} + \frac{\rho_j \phi_{j,\alpha}}{\bar{\rho}_{\alpha}}\right), \qquad \tilde{f}_{j,\alpha+1/2} = \frac{1}{2}\left(f^{+}_{j,\alpha+1/2} + f^{-}_{j,\alpha+1/2}\right).$$

- We get the equality

$$\begin{aligned} G_{\alpha+1/2} = {} & (1-L_\alpha) G_{1/2} + L_\alpha G_{M+1/2} \\ & + \frac{2\bar{\rho}_\alpha \bar{\rho}_{\alpha+1}}{\rho_0(\bar{\rho}_{\alpha+1} + \bar{\rho}_\alpha)} \left((1-L_\alpha) \sum_{\beta=1}^{\alpha} l_\beta \left(\partial_x q_\beta - \sum_{j=1}^{N} \partial_x (r_{j,\beta} u_\beta) \frac{\rho_j - \rho_0}{\rho_j} \right) \right. \\ & \left. - L_\alpha \sum_{\gamma=\alpha+1}^{M} l_\gamma \left(\partial_x q_\gamma - \sum_{j=1}^{N} \partial_x (r_{j,\gamma} u_\gamma) \frac{\rho_j - \rho_0}{\rho_j} \right) + \rho_0 \sum_{j=0}^{N} \tilde{f}_{j,\alpha+1/2} \right). \end{aligned}$$

- Notation: $R_\beta := q_\beta - \sum_{j=1}^{N} r_{j\beta} u_\beta \frac{\rho_j - \rho_0}{\rho_j}$, $\bar{R} := \sum_{\beta=1}^{M} l_\beta R_\beta$. We obtain:

$$\frac{\rho_0(\bar{\rho}_{\alpha+1} + \bar{\rho}_\alpha)}{\bar{\rho}_\alpha \bar{\rho}_{\alpha+1}} G_{\alpha+1/2} - \frac{\rho_0(\bar{\rho}_{\alpha} + \bar{\rho}_{\alpha-1})}{\bar{\rho}_\alpha \bar{\rho}_{\alpha-1}} G_{\alpha-1/2} = l_\alpha \partial_x (R_\alpha - \bar{R}) + \rho_0 \sum_{j=0}^{N} \left(\tilde{f}_{j,\alpha+1/2} - \tilde{f}_{j,\alpha-1/2} \right).$$

Vertical velocity of the mixture

- Let $\alpha \in \{1, \ldots, M\}$. Integrating mass balance eqns over $(z_{\alpha-1/2}, z)$.
- Using the horizontal velocities, the averaged vertical velocities are computed sucessively $\uparrow$:

$$w_{1/2}^{+} = \partial_t z_B + \vec{\boldsymbol{u}}_1 \cdot \nabla_{\boldsymbol{x}} z_B - G_{1/2}/\rho_1.$$

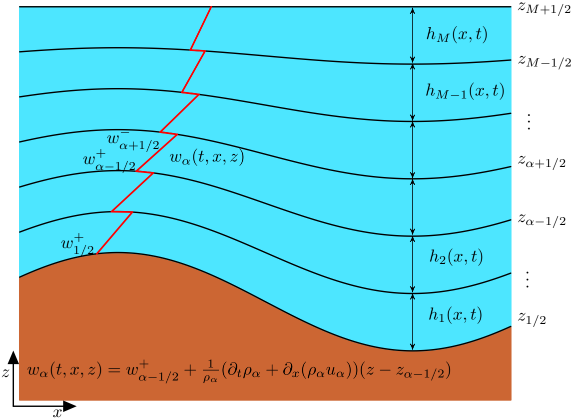
- Then, for $\alpha = 1, \ldots, M$ and $z \in (z_{\alpha-1/2}, z_{\alpha+1/2})$, we set

$$w_\alpha(t, \boldsymbol{x}, z) = w_{\alpha-1/2}^{+} - \frac{1}{\rho_\alpha}\big(\partial_t \rho_\alpha + \nabla_{\boldsymbol{x}} \cdot (\rho_\alpha \vec{\boldsymbol{u}}_\alpha)\big)(z - z_{\alpha-1/2})$$

$$w_{\alpha+1/2}^{-} = w_{\alpha-1/2}^{+} - \frac{h_\alpha}{\rho_\alpha}\big(\partial_t \rho_\alpha + \nabla_{\boldsymbol{x}} \cdot (\rho_\alpha \vec{\boldsymbol{u}}_\alpha)\big),$$

$$w_{\alpha+1/2}^{+} = \frac{1}{\rho_{\alpha+1}}\Big((\rho_{\alpha+1} - \rho_\alpha)\partial_t z_{\alpha+1/2} + (\rho_{\alpha+1}\vec{\boldsymbol{u}}_{\alpha+1} - \rho_\alpha \vec{\boldsymbol{u}}_\alpha) \cdot \nabla_{\boldsymbol{x}} z_{\alpha+1/2} + \rho_\alpha w_{\alpha+1/2}^{-}\Big).$$

Vertical velocity of the mixture



Compact form and eigenvalues

Governing model, compact form :

$$\partial_t \vec{\boldsymbol{W}} + \mathcal{A}(\vec{\boldsymbol{W}})\partial_x \vec{\boldsymbol{W}} = \boldsymbol{G},$$

$\tilde{\boldsymbol{w}} = (\{m_\alpha\}_{\alpha=1}^M, \{q_\alpha\}_{\alpha=1}^M, r_{11}, \ldots, r_{N1}, \ldots, r_{1,M}, \ldots, r_{N,M}),$

$\vec{\boldsymbol{W}} = (\tilde{\boldsymbol{w}}, H),$

$$A(\tilde{\boldsymbol{w}}) = \partial_{\tilde{\boldsymbol{w}}}\mathcal{P}(\tilde{\boldsymbol{w}}) + \mathcal{B}(\tilde{\boldsymbol{w}}),\ \mathcal{A}(\tilde{\boldsymbol{w}}) = \left(\begin{array}{c|c} A(\tilde{\boldsymbol{w}}) & \mathcal{S}(\tilde{\boldsymbol{w}}) \\ \hline 0 & 0 \end{array}\right).$$

- Non conservative products $\mathcal{A}(\vec{\boldsymbol{W}})\vec{\boldsymbol{W}}_x$. Solutions may develop discontinuities and the concept of weak solution in the sense of distributions cannot be used.

G. Dal Maso, P.G. Le Floch, F. Murat, *Definition and weak stability of nonconservative products*, J. Maths. Pures Appl. **74** (1995), 483–548.

Eigenvalues of $\boldsymbol{\mathcal{A}}$

Theorem

If λ_k for $k = 1, \ldots, 2M + NM$ denote the eigenvalues of $\boldsymbol{\mathcal{A}}$ and these are real, then $\bar{u} - \Psi \leq \lambda_k \leq \bar{u} + \Psi$ for all $k = 1, \ldots, 2M + NM$, where

$$\bar{u} := \frac{1}{M}\sum_{\beta=1}^{M} u_\beta, \qquad \Psi := \sqrt{\frac{2M-1}{2M}\left(2\sum_{i=1}^{M}(u_i - \bar{u})^2 + gh\rho_0^{-1}\left(\rho_0 + \frac{1}{M}\sum_{\beta=1}^{M}(2\beta-1)\bar{\rho}_\beta\right)\right)^{1/2}}.$$

If we denote the vector of unknowns as

$$\boldsymbol{w} = (\bar{m}, q_1, \ldots, q_M, r_{1,1}, \ldots, r_{N,1}, \ldots, r_{1,\alpha}, \ldots, r_{N,\alpha}, \ldots, r_{1,M}, \ldots, r_{N,M})^{\mathrm{T}},$$

the system can be written as

$$\partial_t \boldsymbol{w} + \partial_x \boldsymbol{\mathcal{F}}(\boldsymbol{w}) = \boldsymbol{\mathcal{S}}(\boldsymbol{w}, \partial_x \boldsymbol{w}) + \boldsymbol{\mathcal{G}}(\boldsymbol{w}, \partial_x \boldsymbol{w}),$$

$\boldsymbol{\mathcal{F}}(\boldsymbol{w})$, $\boldsymbol{\mathcal{S}}(\boldsymbol{w}, \partial_x \boldsymbol{w})$ and $\boldsymbol{\mathcal{G}}(\boldsymbol{w}, \partial_x \boldsymbol{w})$ are vectors of dimension $M(N+1)+1$:

$$\boldsymbol{\mathcal{F}}(\boldsymbol{w}) = \begin{pmatrix} \sum_{\beta=1}^{M} l_\beta \mathcal{F}^{m_\beta} \\ \boldsymbol{\mathcal{F}}^q \\ \boldsymbol{\mathcal{F}}^{r,1} \\ \vdots \\ \boldsymbol{\mathcal{F}}^{r,M} \end{pmatrix}, \quad \boldsymbol{\mathcal{S}}(\boldsymbol{w}, \partial_x \boldsymbol{w}) = \begin{pmatrix} 0 \\ \boldsymbol{s} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix}, \quad \boldsymbol{\mathcal{G}}(\boldsymbol{w}, \partial_x \boldsymbol{w}) = \begin{pmatrix} 0 \\ \boldsymbol{\mathcal{G}}^q \\ \boldsymbol{\mathcal{G}}^{r,1} \\ \vdots \\ \boldsymbol{\mathcal{G}}^{r,M} \end{pmatrix}.$$

Where:

- The first component of $\boldsymbol{\mathcal{F}}(\boldsymbol{w})$ is defined via $\mathcal{F}^{m_\alpha} = q_\alpha$ for $\alpha = 1, \ldots, M$;
- moreover, $\boldsymbol{\mathcal{F}}^q = (\mathcal{F}^{q_1}, \ldots, \mathcal{F}^{q_M})^{\mathrm{T}}$, where $\mathcal{F}^{q_\alpha} = q_\alpha^2 / m_\alpha$ for $\alpha = 1, \ldots, M$
- and

$$\boldsymbol{\mathcal{F}}^{r,\alpha} := \frac{q_\alpha}{m_\alpha} \begin{pmatrix} r_{1,\alpha} \\ \vdots \\ r_{N,\alpha} \end{pmatrix}, \quad \alpha = 1, \ldots, M.$$

Numerical scheme

If we denote the vector of unknowns as

$$\boldsymbol{w} = (\bar{m}, q_1, \ldots, q_M, r_{1,1}, \ldots, r_{N,1}, \ldots, r_{1,\alpha}, \ldots, r_{N,\alpha}, \ldots, r_{1,M}, \ldots, r_{N,M})^{\mathrm{T}},$$

the system can be written as

$$\partial_t \boldsymbol{w} + \partial_x \boldsymbol{\mathcal{F}}(\boldsymbol{w}) = \boldsymbol{\mathcal{S}}(\boldsymbol{w}, \partial_x \boldsymbol{w}) + \boldsymbol{\mathcal{G}}(\boldsymbol{w}, \partial_x \boldsymbol{w}),$$

$\boldsymbol{\mathcal{F}}(\boldsymbol{w})$, $\boldsymbol{\mathcal{S}}(\boldsymbol{w}, \partial_x \boldsymbol{w})$ and $\boldsymbol{\mathcal{G}}(\boldsymbol{w}, \partial_x \boldsymbol{w})$ are vectors of dimension $M(N+1)+1$:

$$\boldsymbol{\mathcal{F}}(\boldsymbol{w}) = \begin{pmatrix} \sum_{\beta=1}^{M} l_\beta \mathcal{F}^{m_\beta} \\ \boldsymbol{\mathcal{F}}^{q} \\ \boldsymbol{\mathcal{F}}^{r,1} \\ \vdots \\ \boldsymbol{\mathcal{F}}^{r,M} \end{pmatrix}, \quad \boldsymbol{\mathcal{S}}(\boldsymbol{w}, \partial_x \boldsymbol{w}) = \begin{pmatrix} 0 \\ \boldsymbol{s} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix}, \quad \boldsymbol{\mathcal{G}}(\boldsymbol{w}, \partial_x \boldsymbol{w}) = \begin{pmatrix} 0 \\ \boldsymbol{\mathcal{G}}^{q} \\ \boldsymbol{\mathcal{G}}^{r,1} \\ \vdots \\ \boldsymbol{\mathcal{G}}^{r,M} \end{pmatrix}.$$

Where:

- The components of $\boldsymbol{s} = (s_1, \ldots, s_M)^{\mathrm{T}}$ defining the vector $\boldsymbol{\mathcal{S}}$ are given by

$$s_\alpha := g m_\alpha \partial_x (z_b + h) + g h^2 \left(\left(\frac{l_\alpha}{2} + \sum_{\beta=\alpha+1}^{M} l_\beta \right) \partial_x \bar{\rho}_\alpha + \partial_x \left(\sum_{\beta=\alpha+1}^{M} l_\beta (\bar{\rho}_\beta - \bar{\rho}_\alpha) \right) \right)$$

If we denote the vector of unknowns as

$$\boldsymbol{w} = (\bar{m}, q_1, \ldots, q_M, r_{1,1}, \ldots, r_{N,1}, \ldots, r_{1,\alpha}, \ldots, r_{N,\alpha}, \ldots, r_{1,M}, \ldots, r_{N,M})^{\mathrm{T}},$$

the system can be written as

$$\partial_t \boldsymbol{w} + \partial_x \boldsymbol{\mathcal{F}}(\boldsymbol{w}) = \boldsymbol{\mathcal{S}}(\boldsymbol{w}, \partial_x \boldsymbol{w}) + \boldsymbol{\mathcal{G}}(\boldsymbol{w}, \partial_x \boldsymbol{w}),$$

$\boldsymbol{\mathcal{F}}(\boldsymbol{w})$, $\boldsymbol{\mathcal{S}}(\boldsymbol{w}, \partial_x \boldsymbol{w})$ and $\boldsymbol{\mathcal{G}}(\boldsymbol{w}, \partial_x \boldsymbol{w})$ are vectors of dimension $M(N+1)+1$:

$$\boldsymbol{\mathcal{F}}(\boldsymbol{w}) = \begin{pmatrix} \sum_{\beta=1}^{M} l_\beta \mathcal{F}^{m_\beta} \\ \boldsymbol{\mathcal{F}}^q \\ \boldsymbol{\mathcal{F}}^{r,1} \\ \vdots \\ \boldsymbol{\mathcal{F}}^{r,M} \end{pmatrix}, \quad \boldsymbol{\mathcal{S}}(\boldsymbol{w}, \partial_x \boldsymbol{w}) = \begin{pmatrix} 0 \\ \boldsymbol{s} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix}, \quad \boldsymbol{\mathcal{G}}(\boldsymbol{w}, \partial_x \boldsymbol{w}) = \begin{pmatrix} 0 \\ \boldsymbol{\mathcal{G}}^q \\ \boldsymbol{\mathcal{G}}^{r,1} \\ \vdots \\ \boldsymbol{\mathcal{G}}^{r,M} \end{pmatrix}.$$

Where:

- The sub-vectors of $\boldsymbol{\mathcal{G}}$ are defined by $\boldsymbol{\mathcal{G}}^q = (\mathcal{G}^{q_1}, \ldots, \mathcal{G}^{q_M}))^{\mathrm{T}}$ with

$$\mathcal{G}^{q_\alpha} = (\tilde{u}_{\alpha+1/2} G_{\alpha+1/2} - \tilde{u}_{\alpha-1/2} G_{\alpha-1/2})/l_\alpha$$

and

$$\boldsymbol{\mathcal{G}}^{r,\alpha} := \frac{1}{l_\alpha} \left(G_{\alpha+1/2} \tilde{\Phi}_{\alpha+1/2} - G_{\alpha-1/2} \tilde{\Phi}_{\alpha-1/2} - \begin{pmatrix} \rho_1 (\tilde{f}_{1,\alpha+1/2} - \tilde{f}_{1,\alpha-1/2}) \\ \vdots \\ \rho_N (\tilde{f}_{N,\alpha+1/2} - \tilde{f}_{N,\alpha-1/2}) \end{pmatrix} \right)$$

Since we will use the flux function of the unknowns m_α to compute the flux function for the unknown $\bar{m}$, we also consider the part of the source term related to the unknowns m_α, which is defined by

$$\mathcal{G}^{m_\alpha} := (G_{\alpha+1/2} - G_{\alpha-1/2})/l_\alpha, \quad \alpha = 1, \ldots, M.$$

We denote

$$\boldsymbol{w}_\alpha = \begin{pmatrix} m_\alpha \\ q_\alpha \end{pmatrix}, \quad \boldsymbol{\mathcal{F}}_\alpha := \begin{pmatrix} \mathcal{F}^{m_\alpha} \\ \mathcal{F}^{q_\alpha} \end{pmatrix}, \quad \boldsymbol{\mathcal{S}}_\alpha := \begin{pmatrix} 0 \\ s_\alpha \end{pmatrix}, \quad \boldsymbol{\mathcal{G}}_\alpha := \begin{pmatrix} \mathcal{G}^{m_\alpha} \\ \mathcal{G}^{q_\alpha} \end{pmatrix}, \quad \alpha = 1, \ldots, M.$$

Note that using this notation, from the definition of the global system we obtain

$$\partial_t \boldsymbol{w}_\alpha + \partial_x \boldsymbol{\mathcal{F}}_\alpha(\boldsymbol{w}_\alpha) = \boldsymbol{\mathcal{S}}_\alpha + \boldsymbol{\mathcal{G}}_\alpha, \quad \alpha = 1, \ldots, M.$$

The HLL-PVM-1U method is defined by the following two coefficients,

$$\alpha^n_{0,i+1/2} = (S^n_{\mathrm{R},i+1/2}|S^n_{\mathrm{L},i+1/2}| - S^n_{\mathrm{L},i+1/2}|S^n_{\mathrm{R},i+1/2}|)/(S^n_{\mathrm{R},i+1/2} - S^n_{\mathrm{L},i+1/2}),$$

$$\alpha^n_{1,i+1/2} = (|S^n_{\mathrm{R},i+1/2}| - |S^n_{\mathrm{L},i+1/2}|)/(S^n_{\mathrm{R},i+1/2} - S^n_{\mathrm{L},i+1/2}).$$

Here the characteristic velocities $S^n_{\mathrm{L},i+1/2}$ and $S^n_{\mathrm{R},i+1/2}$ are global approximations (they are the same for each layer) of the minimum and maximum wave speed. Taking into account previous Theorem we set the following definition of $S^n_{\mathrm{L},i+1/2}$ and $S^n_{\mathrm{R},i+1/2}$,

$$S^n_{\mathrm{L},i+1/2} = \bar{u}^n_{i+1/2} - \Psi^n_{i+1/2}, \qquad S^n_{\mathrm{R},i+1/2} = \bar{u}^n_{i+1/2} + \Psi^n_{i+1/2}, \tag{5}$$

where

$$\bar{u}^n_{i+1/2} := \frac{1}{M}\sum_{\beta=1}^{M} u^n_{\beta,i+1/2},$$

$$\Psi^n_{i+1/2} := \frac{2M-1}{\sqrt{2M(2M-1)}}\left(2\sum_{\beta=1}^{M}(\bar{u}^n_{i+1/2}-u^n_{i+1/2})^2+\frac{gh^n_{i+1/2}}{\rho_0}\left(\rho_0+\frac{1}{M}\sum_{\beta=1}^{M}(2\beta-1)\bar{\rho}^n_{\beta,i+1/2}\right)\right)^{1/2}$$

where M is the number of layers.

HLL-PVM-1U method

The HLL-PVM-1U method proposed can be written as

$$\boldsymbol{w}_{\alpha,i}^{n+1} = \boldsymbol{w}_{\alpha,i}^{n} - \frac{\Delta t}{\Delta x}\left(\tilde{\boldsymbol{\mathcal{F}}}_{\alpha,i+1/2}^{n} - \tilde{\boldsymbol{\mathcal{F}}}_{\alpha,i-1/2}^{n}\right) + \Delta t \boldsymbol{\mathcal{S}}_{\alpha,i}^{n} + \Delta t \boldsymbol{\mathcal{G}}_{\alpha,i}^{n},$$

where here the numerical flux is given by $\tilde{\boldsymbol{\mathcal{F}}}_{\alpha,i+1/2}^{n} = (\tilde{\mathcal{F}}_{i+1/2}^{m_\alpha,n}, \tilde{\mathcal{F}}_{i+1/2}^{q_\alpha,n})^{\mathrm{T}}$,

$$\tilde{\boldsymbol{\mathcal{F}}}_{\alpha,i+1/2}^{n} = \frac{1}{2}\left(\boldsymbol{\mathcal{F}}_{\alpha}\left(\boldsymbol{w}_{\alpha,i+1}^{n}\right) + \boldsymbol{\mathcal{F}}_{\alpha}\left(\boldsymbol{w}_{\alpha,i}^{n}\right)\right) - \frac{1}{2}\Big(\alpha_{0,i+1/2}^{n}\left(\boldsymbol{w}_{\alpha,i+1}^{n} - \boldsymbol{w}_{\alpha,i}^{n} + \boldsymbol{\mathcal{C}}_{\alpha,i+1/2}^{n} + \boldsymbol{S}_{\alpha,i+1/2}^{n}\right)$$
$$+ \alpha_{1,i+1/2}^{n}\left(\boldsymbol{\mathcal{F}}_{\alpha}\left(\boldsymbol{w}_{\alpha,i+1}^{n}\right) - \boldsymbol{\mathcal{F}}_{\alpha}\left(\boldsymbol{w}_{\alpha,i}^{n}\right) + \boldsymbol{S}_{\alpha,i+1/2}^{n}\right)\Big),$$

where

$$\boldsymbol{\mathcal{C}}_{\alpha,i+1/2}^{n} = \begin{pmatrix} \dfrac{\bar{\rho}_{\alpha,i+1}^{n} + \bar{\rho}_{\alpha,i}^{n}}{2}(z_{i+1} - z_i) \\ 0 \end{pmatrix}, \quad \boldsymbol{S}_{\alpha,i+1/2}^{n} = g\begin{pmatrix} 0 \\ s_{\alpha,i+1/2}^{n} \end{pmatrix},$$

$$s_{\alpha,i+1/2}^{n} = \frac{1}{2}\Bigg((m_{i+1}^{n} + m_i^{n})(\eta_{i+1}^{n} - \eta_i^{n}) + (h_{i+1}^{2,n} + h_i^{2,n})\left(\frac{l_\alpha}{2} + \sum_{\beta=\alpha+1}^{M} l_\beta\right)(\bar{\rho}_{\alpha,i+1}^{n} - \bar{\rho}_{\alpha,i}^{n})$$
$$+ (h_{i+1}^{n} + h_i^{n})\sum_{\beta=\alpha+1}^{M} l_\beta\left((\bar{\rho}_{\beta,i+1}^{n} - \bar{\rho}_{\alpha,i+1}^{n})h_{i+1}^{n} - (\bar{\rho}_{\beta,i}^{n} - \bar{\rho}_{\alpha,i}^{n})h_i^{n}\right)\Bigg),$$

and $\boldsymbol{\mathcal{G}}_{\alpha,i}^{n} = \begin{pmatrix} \mathcal{G}_i^{m_\alpha,n} \\ \mathcal{G}_i^{q_\alpha,n} \end{pmatrix}$.

Since the solid concentrations are passive scalars in the system, i.e. $\mathcal{F}^{r_{j,\alpha}} = (r_{j,\alpha}/m_\alpha)\mathcal{F}^{m_\alpha}$, we use the following upwinding formula to compute the numerical flux relative to $r^n_{j,\alpha}$:

$$\tilde{\mathcal{F}}^{r_{j,\alpha},n}_{i+1/2} = \begin{cases} (r^n_{j,\alpha,i}/m^n_{\alpha,i})\tilde{\mathcal{F}}^{m_\alpha,n}_{i+1/2} & \text{if } \tilde{\mathcal{F}}^{m_\alpha,n}_{i+1/2} > 0, \\ (r^n_{j,\alpha,i+1}/m^n_{\alpha,i+1})\tilde{\mathcal{F}}^{m_\alpha,n}_{i+1/2} & \text{otherwise,} \end{cases} \qquad j = 1, \dots, N.$$

Finally, the numerical scheme to approximate the unknowns of the problem is defined as follows:

$$\bar{m}_i^{n+1} = \bar{m}_i^n - \frac{\Delta t}{\Delta x}\sum_{\beta=1}^{M} l_\beta \tilde{\mathcal{F}}_{i+1/2}^{m_\beta,n},$$

$$q_{\alpha,i}^{n+1} = q_{\alpha,i}^n - \frac{\Delta t}{\Delta x}\left(\tilde{\mathcal{F}}_{i+1/2}^{q_\alpha,n} - \tilde{\mathcal{F}}_{i-1/2}^{q_\alpha,n}\right) + \frac{\Delta t}{2}\left(s_{\alpha,i+1/2}^n + s_{\alpha,i-1/2}^n\right) + \Delta t \mathcal{G}_i^{q_\alpha,n},$$

$$r_{j,\alpha,i}^{n+1} = r_{j,\alpha,i}^n - \frac{\Delta t}{\Delta x}\left(\tilde{\mathcal{F}}_{i+1/2}^{r_{j,\alpha},n} - \tilde{\mathcal{F}}_{i-1/2}^{r_{j,\alpha},n}\right) + \Delta t \mathcal{G}_i^{r_{j,\alpha},n},$$

with

$$\mathcal{G}_i^{r_{j,\alpha},n} = \frac{1}{l_\alpha}\left(\bar{\phi}_{j,\alpha+1/2,i}^n G_{\alpha+1/2,i}^n - \bar{\phi}_{j,\alpha-1/2,i}^n G_{\alpha-1/2,i}^n\right) - \frac{\rho_j}{l_\alpha}\left(\hat{f}_{j,\alpha+1/2,i+1/2}^n - \hat{f}_{j,\alpha-1/2,i+1/2}^n\right).$$

Numerical tests:

- $g = 9.8\,\mathrm{m/s^2}$ (acceleration of gravity), $\phi_{\max} = 0.68$, $n_{\mathrm{RZ}} = 4.7$, $\mu_0 = 0.02416\,\mathrm{Pa\,s}$, $\rho_0 = 1208\,\mathrm{kg/m^3}$, $\rho_1 = \cdots = \rho_N = 2790\,\mathrm{kg/m^3}$.
- CFL cond to determine Δt in each iteration:

$$\frac{\Delta t}{\Delta x} \max_{1 \le i \le C} \max\{|S_{\mathrm{R},i+1/2}|, |S_{\mathrm{L},i+1/2}|\} = \mathrm{CFL},$$

  where $S_{\mathrm{R},i+1/2}$ and $S_{\mathrm{L},i+1/2}$ are the bounds of eigenvalues, $\mathrm{CFL} = 0.5$.

Test 1: 1D vertical sedimentation, $N = 3$

- $d_1 = 4.96 \times 10^{-4}\,\mathrm{m}$, $d_2 = 3.25 \times 10^{-4}\,\mathrm{m}$, $d_3 = 10^{-4}\,\mathrm{m}$, $h = 0.3\,\mathrm{m}$, $M = 50$, $\phi_1(t=0) = 0.1$, $\phi_2(t=0) = 0.05$, $\phi_3(t=0) = 0.09$

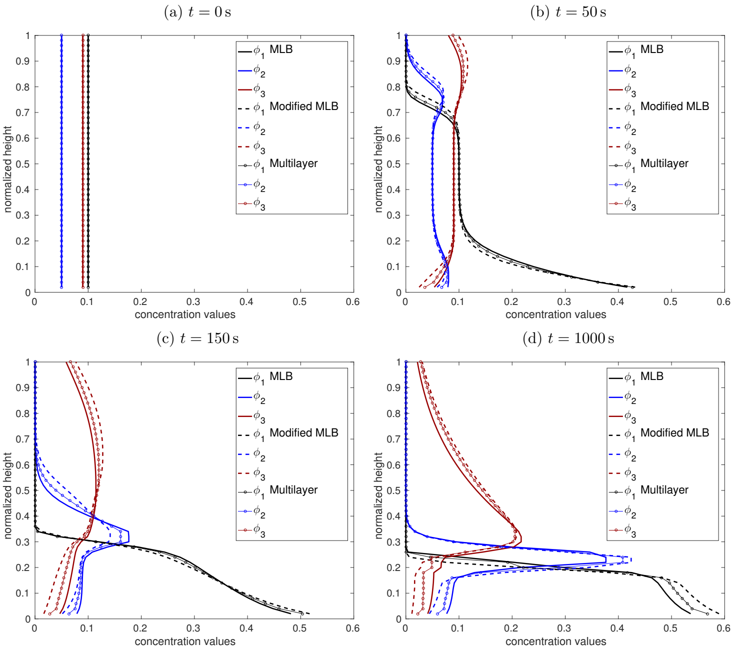


Figure: Test1: 1D vertical sedimentation.

Test 2: channel with inclined bottom

Test 2

- channel of length $L = 1\,\text{m}$, $N = 2$, $\rho_0 = 1208\,\text{kg}/\text{m}^3$, $d_1 = 4.96 \times 10^{-4}\,\text{m}$, $d_2 = 1.25 \times 10^{-4}\,\text{m}$,

$$z_B(x) = -0.1x + 0.1\,\text{m} \quad x \in [0, L].$$

- Initial condition

$$\phi_{1,\alpha}(0,x) = 0, \quad \phi_{2,\alpha}(0,x) = 0, \quad u_\alpha(0,x) = 0,$$

and for the height $h(t = 0) = 0.3 - z_B$.

- boundary condition: linear horizontal velocity, average $0.15\,\text{m}/\text{s}$, $u(z)_{|x=0} = 0.133z + 0.128\,\text{m}/\text{s}$,

$$\sum_{\alpha=1}^{M} \phi_{1,\alpha|x=0} = 0.05, \sum_{\alpha=1}^{M} \phi_{2,\alpha|x=0} = 0.025.$$

right bound: homogeneous Neumann condition, $M = 10$ layers

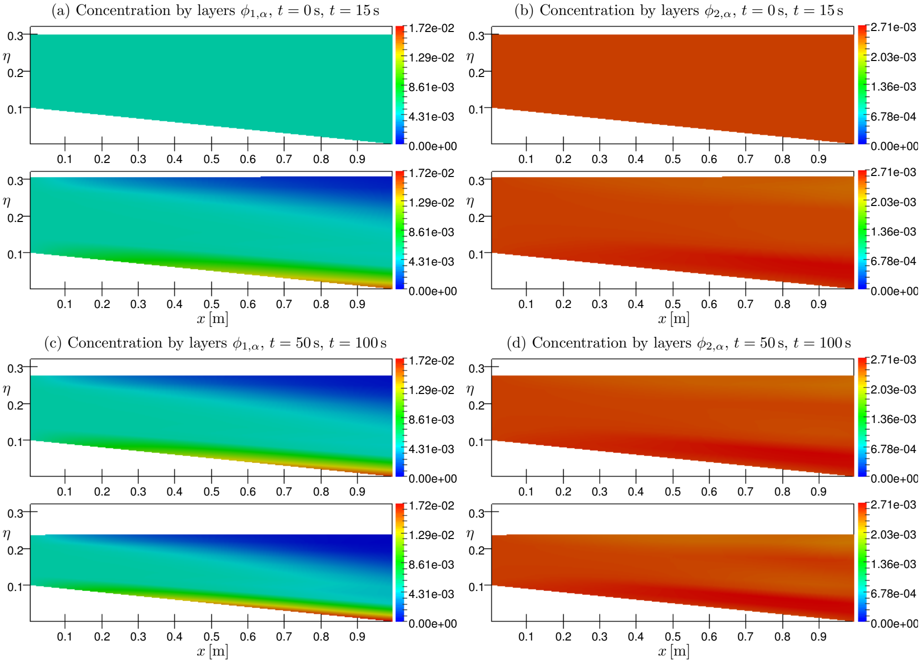


Figure: Test2: Imposed velocity.

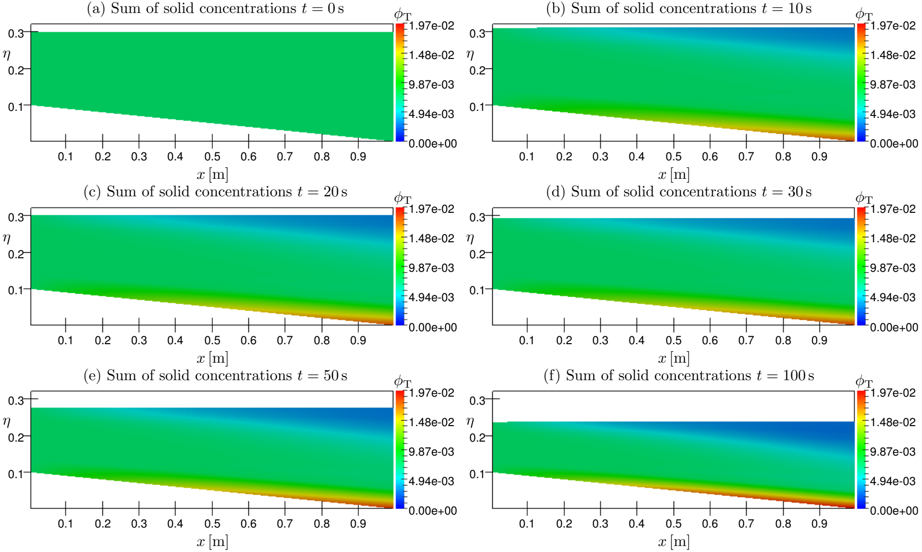


Figure: Test2: Concentration by color by $\phi_T = \phi_1 + \phi_2$, $\eta(x) = z_B(x) + h(x)$ m. .

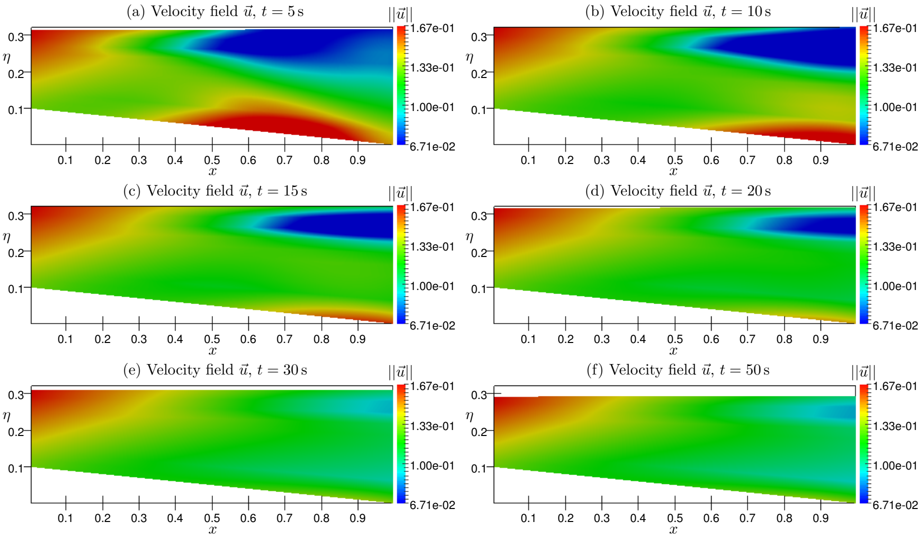


Figure: Test2: Magnitude of the velocity field $\vec{u}$.

Test 3

- Same mixture as before. The bottom elevation is given by

$$z_B(x) = 0.2\exp(-40(x-0.5)^2), \quad x \in [0, L],$$

and the initial condition for this test is given by

$$\sum_{\alpha=1}^{M} \phi_{1,\alpha}(0,x) = 0.05, \quad \sum_{\alpha=1}^{M} \phi_{2,\alpha}(0,x) = 0.025,$$
$$u_\alpha(0,x) = 0 \quad \text{for all } \alpha = 1,\dots,M, \quad \text{for all } x \in [0,L],$$

zero-flux boundary conditions, $M = 10$ layers, 150 horizontal cells

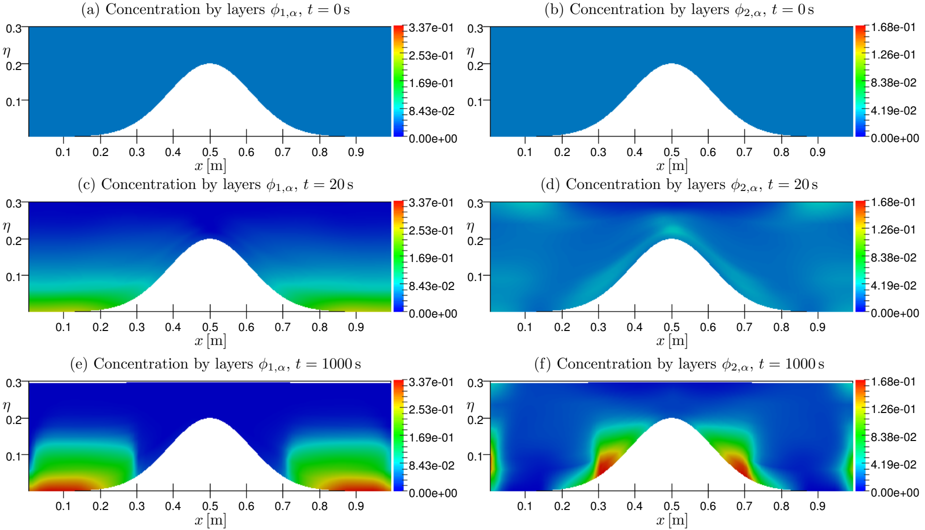


Figure: Test3: Concentration of ϕ_1 and ϕ_2 by color in a domain with a bump, $\eta(x) = z_B(x) + h(x)$ m.

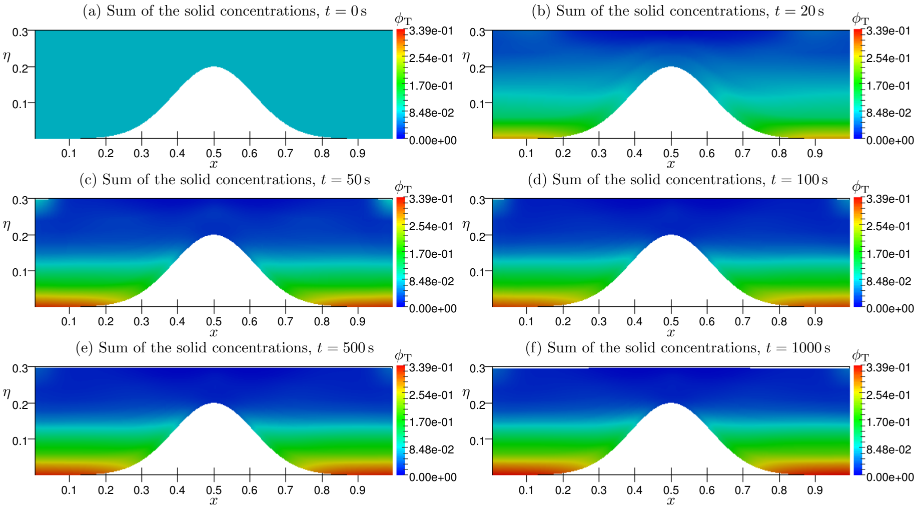


Figure: Test3: Concentration by color by $\phi_T = \phi_1 + \phi_2, \eta(x) = z_B(x) + h(x)$ m.

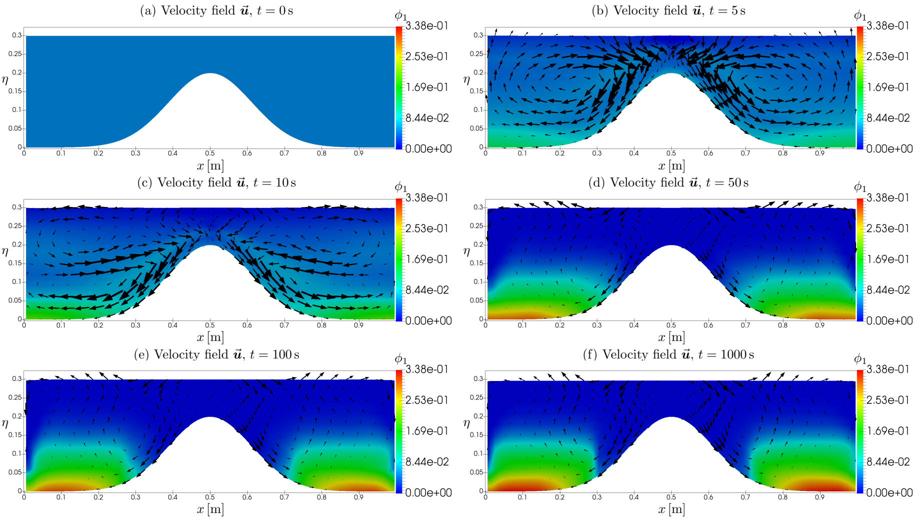


Figure: Test3: Velocity field $\vec{u}$ over concentration ϕ_1, $\eta(x) = z_B(x) + h(x)$ m.

Concluding remarks

- ML SW model can be used for simulations in industrial applications, but is especially suitable for natural geophysical processes such as sediment transport and polydisperse sedimentation in rivers and estuaries.
- Model provides the velocity field of the mixture, the concentrations of the each solid species, and the evolution of the free surface.
- Currently implementing an extension of the scheme to two horizontal space dimensions, including viscous and compression terms.
- Simulating further scenarios such as gravity currents of interest.

Thanks for your attention.

Concluding remarks

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Thanks for your attention.

Settling velocities definition for global mass conservation of polydisperse sedimentation models

E.D. Fernández-Nieto

Universidad de Sevilla
http://personal.us.es/edofer

A joint work with R. Bürger and V. Osores (U. Concepción, Chile)

Balance laws in fluid mechanics, geophysics, biology
(theory, computation and application)
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