High order methods for balance laws, with an application to gas dynamics with gravity force

Gabriella Puppo, Matteo Semplice, ...

Dipartimento di Scienza e Alta Tecnologia
Università dell'Insubria
http://gabriellapuppo.it

Balance laws in fluid mechanics, geophysics, biology
(theory, computation, and application)
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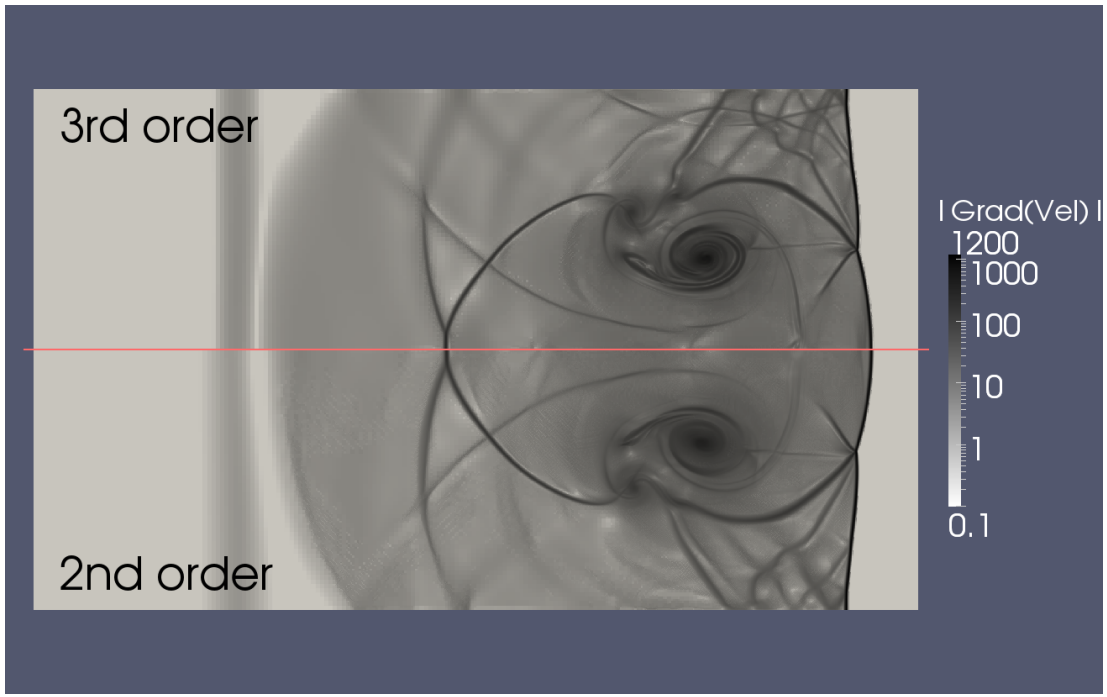
Central WENO



(from Coco, Russo, Semplice, JSC 2016)

Finite volume methods

Consider a hyperbolic system of balance laws of the form

$$\partial_t \boldsymbol{u} + \nabla_x \cdot \boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{s}(\boldsymbol{u}).$$

To integrate the system, one covers the computational domain with N elements $\Omega_j, j = 1, \ldots, N$. Define the cell average of the unknown

$$\boldsymbol{u}_j = \frac{1}{|\Omega_j|} \int_{\Omega_j} \boldsymbol{u} \, \mathrm{d}x.$$

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Integrating the PDEs on each element, one finds the evolution equation for the cell averages as

$$\frac{\mathrm{d}\boldsymbol{u}_j}{\mathrm{d}t} = -\frac{1}{|\Omega_j|} \int_{\partial\Omega_j} \boldsymbol{f} \cdot \boldsymbol{n} \, \mathrm{d}s + \frac{1}{|\Omega_j|} \int_{\Omega_j} \boldsymbol{s}(\boldsymbol{u}) \, \mathrm{d}x.$$

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- Quadrature rules to approximate the line and volume integrals.
- High order reconstruction algorithm, to estimate the point values of $\boldsymbol{u}$ along $\partial\Omega_j$, and within Ω_j, from the cell averages.
- Approximation of the fluxes along $\partial\Omega_j$ accounting for intercell communication (approximate Riemann solvers).
- Approximate integration in time.

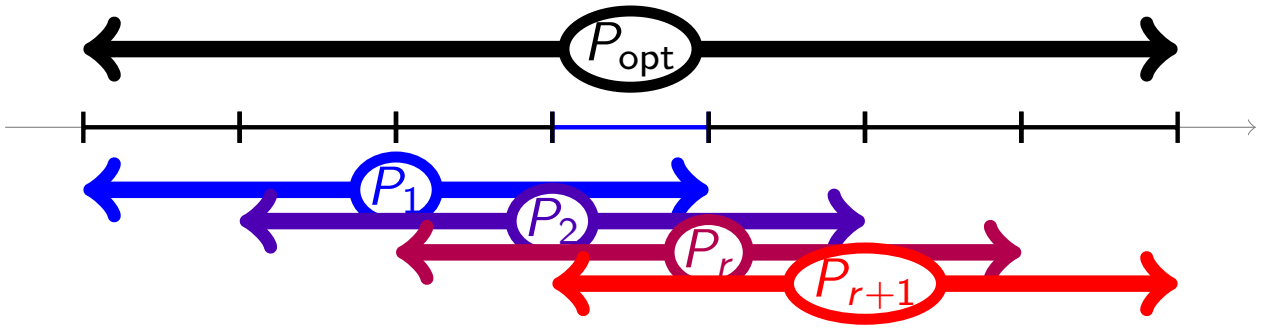
Reconstructions

The key point in finite volume schemes is the reconstruction, which provides from the cell averages $\boldsymbol{u}_j$ the point values along the boundary of Ω_j, and at the interior quadrature nodes.
The reconstruction must be

- fast to compute: use polynomials to approximate the data;
- high order accurate: choose a high degree interpolation polynomial, which is based on a stencil, i.e. a set of cells around the cell Ω_j;
- non oscillatory: choose only information coming from cells which do not contain discontinuities: non linear algorithm;
- efficient: recycle computations as much as possible.

Weighted essentially non-oscillatory reconstructions (1D)

Given the cell averages $\overline{u}_{j-r}, \ldots, \overline{u}_{j+r}$ of a bounded function $u(x)$,



$$(P_{\text{opt}})_j \text{ s.t. } \forall i = -r, \ldots, r: \qquad \frac{1}{|\Omega_{j+i}|} \int_{\Omega_{j+i}} P_{\text{opt}}(x)\mathrm{d}x = \overline{u}_{j+i}$$

- If $\mathcal{R}_j = (P_{\text{opt}})_j$, the accuracy is $O(h^{2r+1})$ in smooth regions.
- However $(P_{\text{opt}})_j$ is oscillatory if a discontinuity is present in its stencil.
- Thus, downgrade, if needed, to a lower accuracy non-oscillatory alternative, $\mathcal{R}_j = P_k$, s.t. P_k contains no discontinuities[1].

[1] Shu, 1997

Example: WENO3 reconstruction in 1D

Third order linear reconstruction algorithm: $\mathcal{R}(x)$

- stencil of 3 cells: $\Omega_{j-1}, \Omega_j, \Omega_{j+1}$;
- $\exists! P_{\text{opt}} \in \mathbb{P}_2 : \int_{\Omega_i} P_{\text{opt}} \mathrm{d}x = |\Omega_i| \overline{u}_i$ for $i = j-1, j, j+1$.

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- Choosing $\mathcal{R} = P_{\text{opt}}$ would be
  - third order accurate on smooth data,
  - oscillatory in the presence of discontinuities.

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Instead, for each reconstruction point ξ:

- consider $P_L \in \mathbb{P}_1$ interpolating $\overline{u}_j$ and $\overline{u}_{j-1}$;
- consider $P_R \in \mathbb{P}_1$ interpolating $\overline{u}_j$ and $\overline{u}_{j+1}$;
- find a convex combination:
  $P_{\text{opt}}(\xi) = d_L(\xi) P_L(\xi) + d_R(\xi) P_R(\xi)$;

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  $P_{\text{opt}}(\xi) = d_L(\xi) P_L(\xi) + d_R(\xi) P_R(\xi)$;
- compute nonlinear weights ω_L and ω_R such that

  $\Rightarrow$ on smooth data: $\omega_j \approx d_j$ and $\mathcal{R}_j(\xi) \approx P_{\text{opt}}(\xi)$

  $\Rightarrow$ otherwise either $\omega_R \approx 0$ and $\mathcal{R}_j(\xi) \approx P_L(\xi)$
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    - or $\omega_L \approx 0$ and $\mathcal{R}_j(\xi) \approx P_R(\xi)$;
- set $\mathcal{R}_j(\xi) := \omega_L(\xi) P_L(\xi) + \omega_R(\xi) P_R(\xi)$

Summing up

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- Since $\mathcal{R} = P_{\text{opt}}$ when the flow is smooth, the reconstruction algorithm becomes linear on smooth flows.
- The presence of discontinuities triggers the non linearities of the scheme, choosing lower degree polynomials, based on smooth stencils.

The pain of several reconstruction points

For a FV scheme in 2D, several reconstruction points are needed to update a single cell. With WENO, the reconstruction must be repeated at each point.

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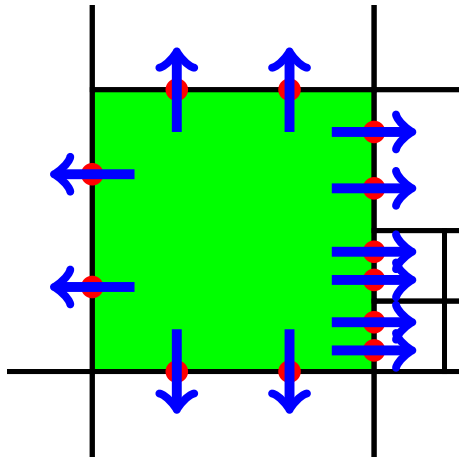

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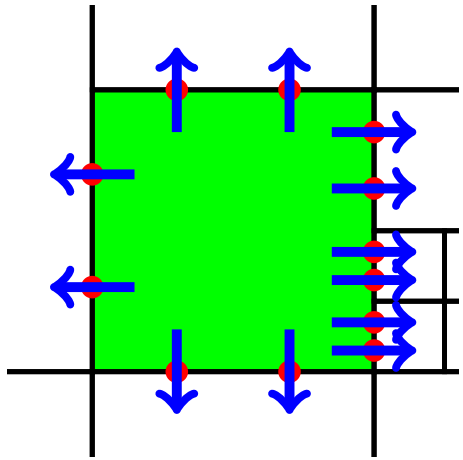

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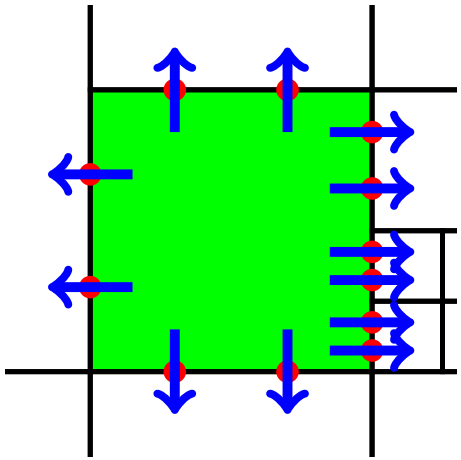

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You need a reconstruction which is not based on a single point.

A single reconstruction for all points

Recall, WENO3:

$$\text{Given } \hat{x} \in \Omega,\ \mathcal{R}(\hat{x}) = d_L(\hat{x})P_L(\hat{x}) + d_R(\hat{x})P_R(\hat{x}) \qquad \text{(WENO3)}$$

Levy, Puppo, Russo, M2AN (1999)

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~~Given $\hat{x} \in \Omega$, $\mathcal{R}(\hat{x}) = d_L(\hat{x})P_L(\hat{x}) + d_R(\hat{x})P_R(\hat{x})$~~ (WENO3)

is replaced by

$$\boxed{\forall x : \mathcal{R}(x) = d_0\, P_0(x) + d_L P_L(x) + d_R P_R(x)} \quad \text{(CWENO3)}$$

how?

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why? d_k do not depend on the reconstruction point

$\Rightarrow$ no dependence on mesh topology,
not even in 2d/3d, AMR, ...

CWENO, the general case

Let $p = 2r + 1$ be the required accuracy, where r is the degree of the $r + 1$ low order polynomials P_k forming the standard WENO reconstruction. Now,

1. choose $d_0, d_1, d_{r+1} \in (0, 1)$ such that $\sum_{k=0}^{r+1} d_k = 1$;
2. compute $P_0(x) := \frac{1}{d_0}\left(P_{\text{opt}}(x) - \sum_{k=1}^{r+1} d_k P_k(x)\right)$;
3. compute WENO-style nonlinear weights $d_k \rightsquigarrow \omega_k$; (no x dependence!)
4. compute the reconstruction polynomial (unif. accurate in the cell!)

$$\mathcal{R}(x) = \sum_{k=0}^{r+1} \omega_k P_k(x) \; = u(x) + O(h)^p; \qquad \forall x \in \text{cell}$$

5. evaluate $\mathcal{R}(x)$ on each reconstruction point needed.

Cravero, P., Semplice, Visconti Math. Comp. (2018)

Background

- Shu, C.W. Lecture Notes in Math., Springer, (1998).
- Levy, P., Russo M2AN (1999)
- Capdeville JCP (2008)
- Aràndiga, Baeza, Belda, Mulet. SINUM (2011)
- Coco, Russo, Semplice, J. Sci. Comp. (2016)
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Spectral properties

A more efficient reconstruction, but...

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- This brought us to introduce the new concepts of distortion and temperature for a numerical scheme for conservation laws.

Von Neumann analysis

Consider the linear advection equation $u_t + a u_x = 0$, with periodic initial and boundary conditions on $(0, 2\pi)$.

- The evolution of a single Fourier mode $u_k(x,t) = \hat{u}_k(t)\exp(ikx)$ is given by

$$\frac{\mathrm{d}\hat{u}_k}{\mathrm{d}t} e^{ikx} = -ik\, a\, \hat{u}_k(t)\ e^{ikx}, \qquad u(x, t=0) = u_0(x).$$

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- Then the exact solution can be written as

$$u(x,t) = \sum_k \hat{u}_k(0) e^{ik(x-at)}, \qquad \hat{u}_k(0) = \frac{1}{2\pi}\int_0^{2\pi} u_0(x) e^{-ikx}.$$

Von Neumann analysis

Solving the same equation with a linear finite difference scheme on the stencil $\{x_{\ell h}\}, \ell = -r \dots r$, for a single Fourier mode $u_k(x,t) = \hat{u}_k(t) e^{ikx}$ yields

$$\frac{\mathrm{d}\hat{u}_k}{\mathrm{d}t} e^{ikx} = -a\, \hat{u}_k(t) D_x(e^{ikx}),$$

and the discrete derivative D_x is given by

$$D_x(e^{ikx}) = \left(\sum_{\ell=-r}^{r} c_\ell e^{ikh\ell} \right) e^{ikx} = (ik + \tilde{\omega}_k)\, e^{ikx}.$$

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- So e^{ikx} is an eigenfunction also for the discrete derivative D_x, except that the amplitude of a single Fourier mode is modified to

$$u_k(x,t) = \hat{u}_k(0) e^{ik(x-at)} e^{-a\tilde{\omega}_k t}.$$

Thus the quantity $\tilde{\omega}_k$ measures the spurious effects due to the discrete approximation, with $\tilde{\omega}_k \approx O(h^p)$.

Artificial diffusion

The real part of $\tilde{\omega}_k$ induces a spurious damping of the amplitude of $u_k(x,t)$, which is faster for high frequency modes ($k >> 1$).
This is called numerical diffusion: the small scale modes tend to disappear.

For first order Upwind

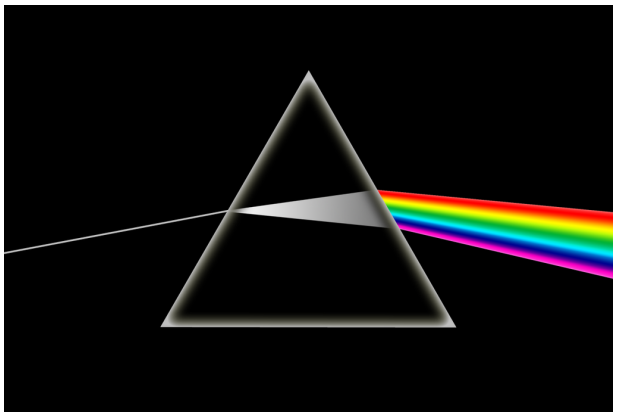
$$\tilde{\omega}_k = -\tfrac{1}{2}k^2 h + O(h^2)$$

and

$$u_k(x,t) \approx \hat{u}_k(0) e^{ik(x-at)} e^{-\frac{1}{2}ak^2 ht}$$

Artificial dispersion

The imaginary part of $\tilde{\omega}_k$ induces a spurious propagation speed. Each mode $u_k(x,t)$ moves with speed $\tilde{a} = a + \frac{a}{k}\mathrm{Im}(\tilde{\omega}_k)$. Again, this effect is stronger for high frequency modes ($k >> 1$).
This is called numerical dispersion: the small scale modes tend to move with high relative speed with respect to the initial wave packet. Thus the Fourier modes separate, and the solution becomes oscillatory.



For a second order scheme

$$\tilde{\omega}_k = -\tfrac{1}{6} i k^3 h^2 + O(h^3)$$

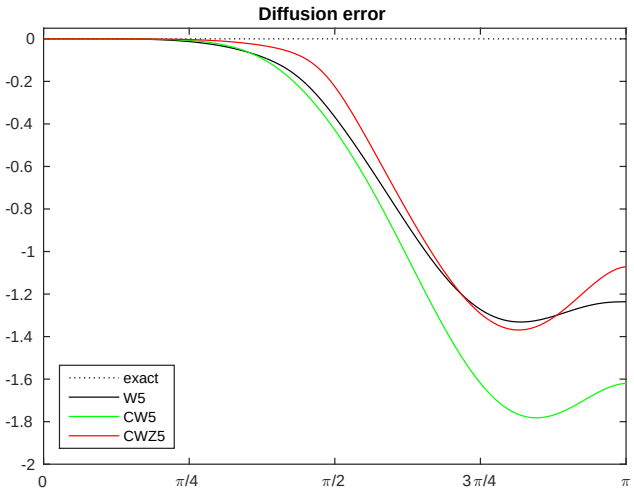
and

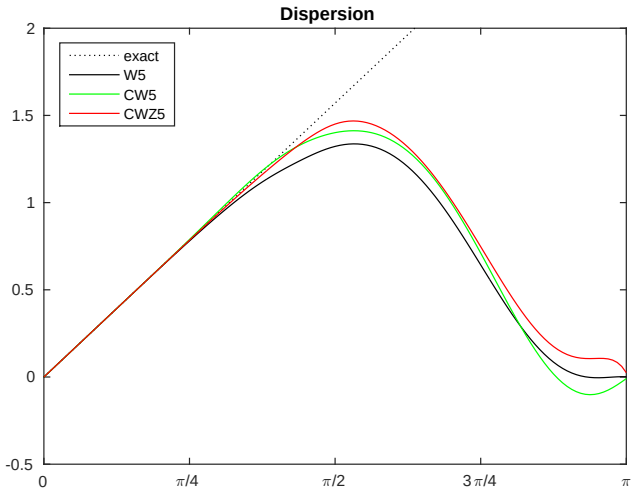
$$u_k(x,t) \approx \hat{u}_k(0) e^{ik(x - a(1 - \frac{1}{6}h^2k^2)t)}.$$

Pirozzoli JCP (2006)

Diffusion and dispersion for WENO and CWENO

$\mathrm{Re}(\widetilde{\omega_k})$ and $\mathrm{Im}(\widetilde{\omega_k})$, as a function of $\ell = \pi k/N$ for WENO (black), CWENO (green) and the modified version CWENOZ (red). Order 5.





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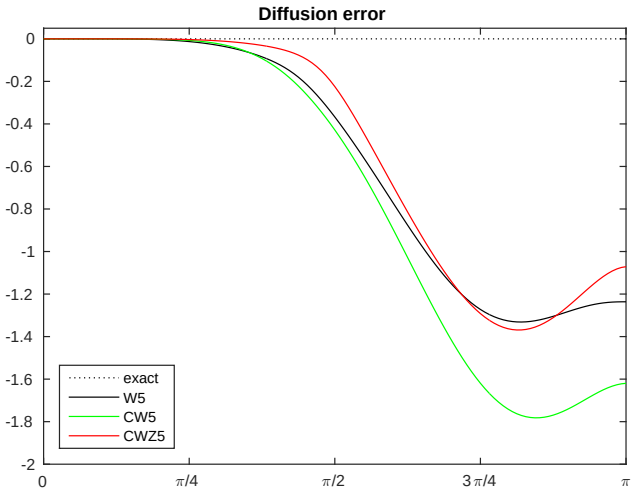

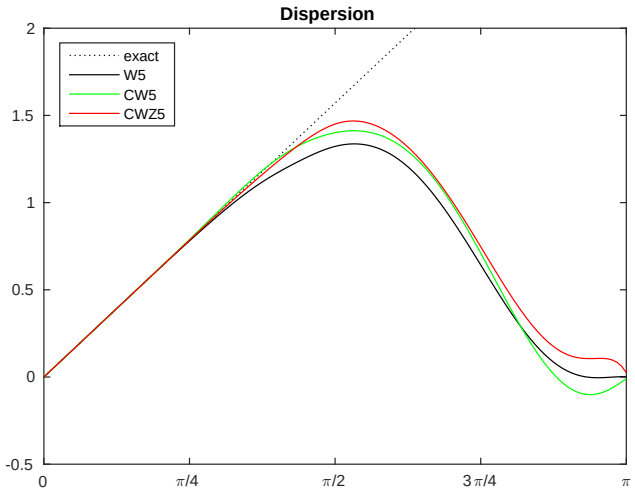

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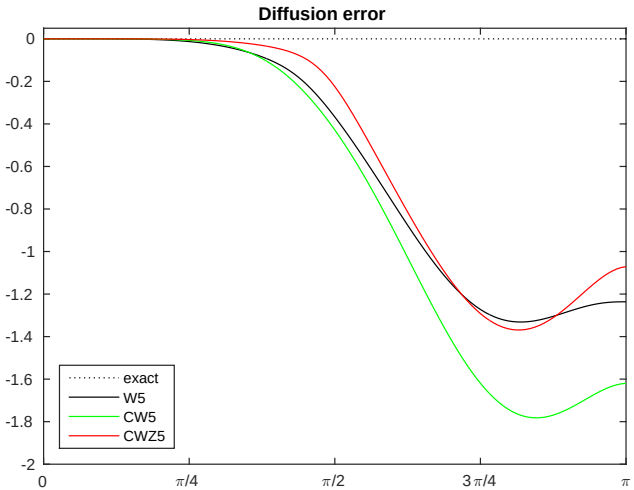
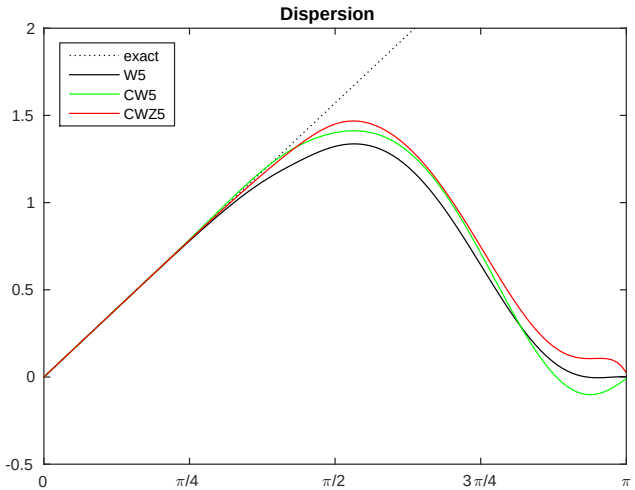




- Clearly, for $\ell > \pi/2$, no scheme can resolve the waves correctly: one has less than 2 grid points per wave number.
- All schemes are comparable, but with a definite edge for CWENOZ.

Diffusion and dispersion in the non linear case

In the non linear case, Fourier modes are coupled. But still one can study the effect of the numerical derivative on each mode $D_x e^{ikx}$. Since we are working on real functions, let

$$D_x \begin{bmatrix} \sin(kx) \\ \cos(kx) \end{bmatrix} = \sum_{\ell=1}^{N} \begin{bmatrix} \omega_{2\ell,2k} & \omega_{2\ell,2k+1} \\ \omega_{2\ell+1,2k} & \omega_{2\ell+1,2k+1} \end{bmatrix} \begin{bmatrix} \sin(\ell x) \\ \cos(\ell x) \end{bmatrix},$$

This defines a matrix Ω. The exact derivative is

$$\mathbb{D} = \operatorname{diag}\left(k \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right), \qquad k = 1, \dots, N.$$

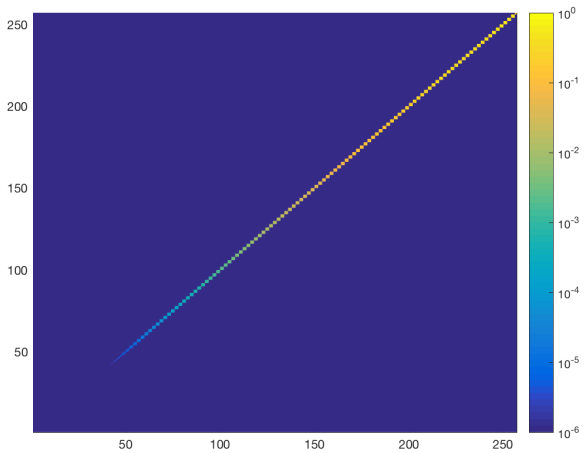
Thus $\mathbb{E} = \Omega - \mathbb{D}$ defines the error matrix.

Cravero, P., Semplice, Visconti Comp. Fluids (2018)

Diffusion, dispersion and distortion

With the introduction of the error matrix $\mathbb{E}$, we could extend previous analysis of non linear schemes.

If the scheme is linear, the matrix $\mathbb{E}$ is block-diagonal with 2×2 blocks along the diagonal. These blocks contain the artificial diffusion and dispersion information of the scheme.

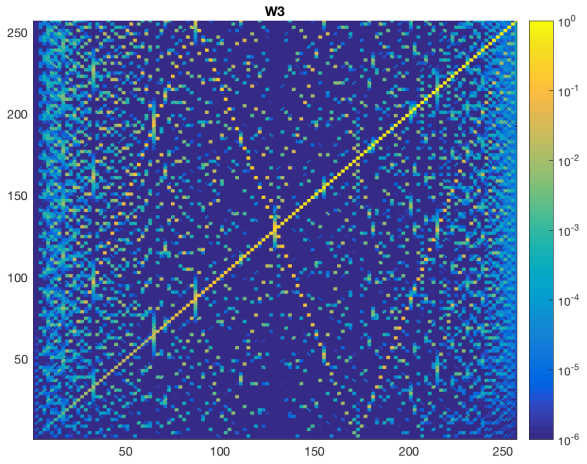


Diffusion, dispersion and distortion

With the introduction of the error matrix $\mathbb{E}$, we could extend previous analysis of non linear schemes.

If the scheme is non linear, still the 2×2 blocks along the diagonal give information on how the k-th mode is transformed. But now there are non-zero terms also away from the main diagonals: the size of these terms measures distorsive effects

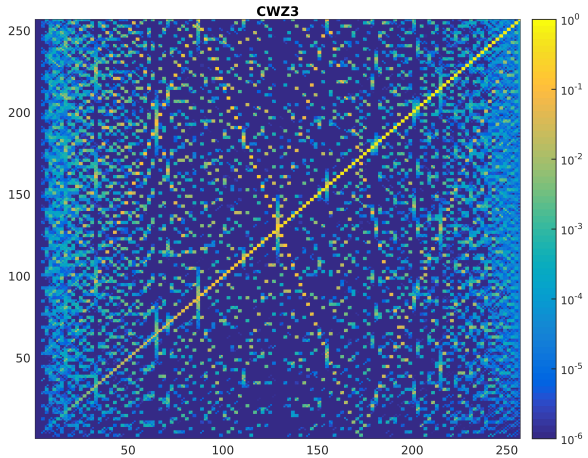
WENO3



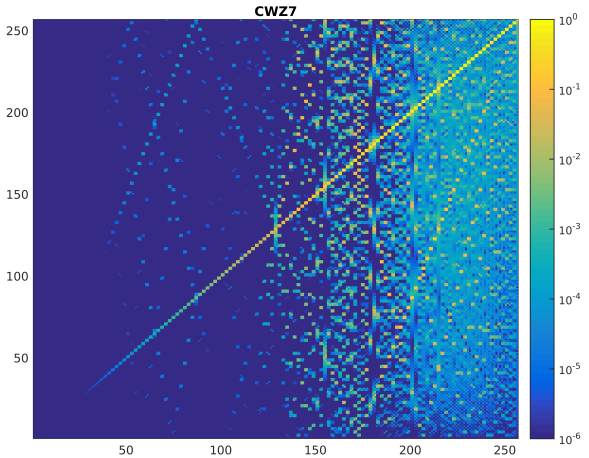
Distortion for CWENO schemes

The amplitude of the coefficients of the error matrix $\mathbb{E}$ shows that as the order is increased, distorsive effects decrease. .

CWENO3



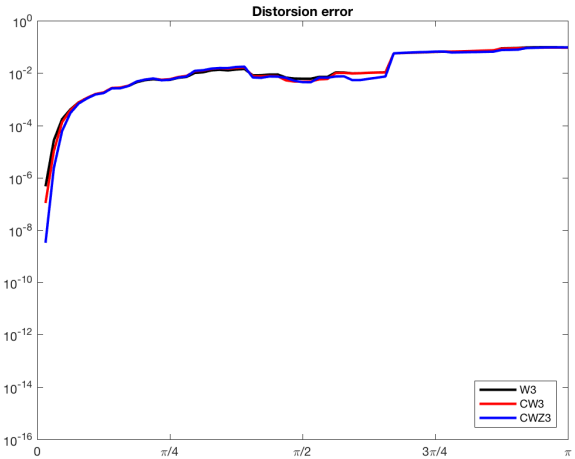
CWENO7



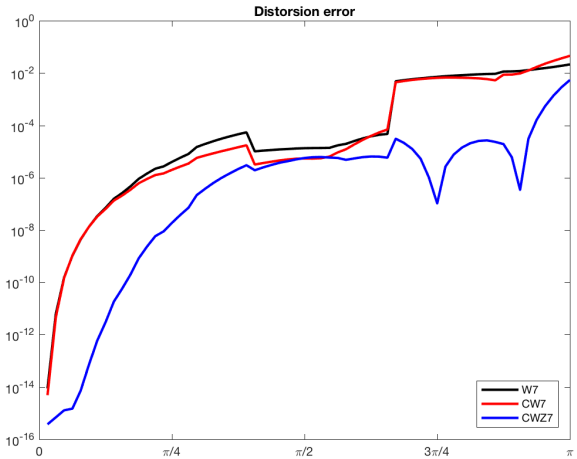
Comparing different high order schemes

We study the distortion errors of CWENO and WENO schemes for different orders of accuracy.

3rd order



7th order



WENO (black), CWENO (red), CWENOZ (blue).

Temperature

The size of the spurious modes determines the distortion of a scheme, but another interesting parameter is also how far, in frequency space, are the spurious modes from the exact mode.
We quantify this idea with the notion of Temperature on the k-th mode

$$T_k = \frac{1}{N^3} \sum_{\ell=1}^{N} (\Omega_{\mathbb{C}})_{\ell k} \left(\frac{k-\ell}{\pi} \right)^2 .$$

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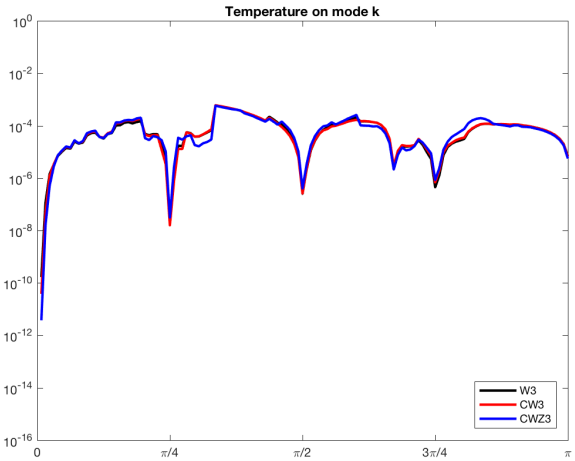
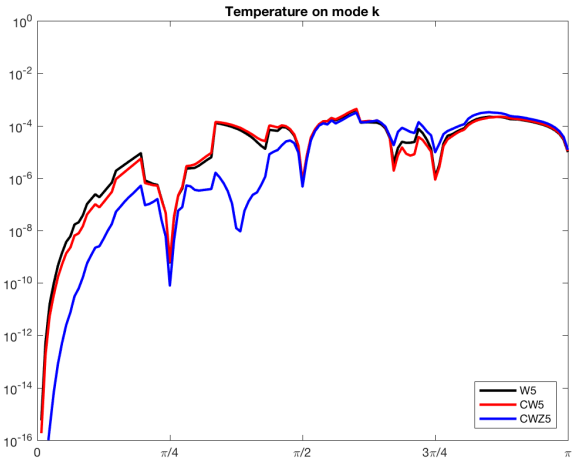


Order 5



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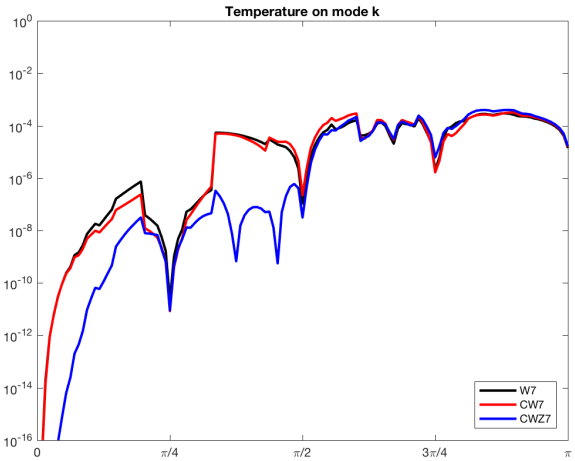
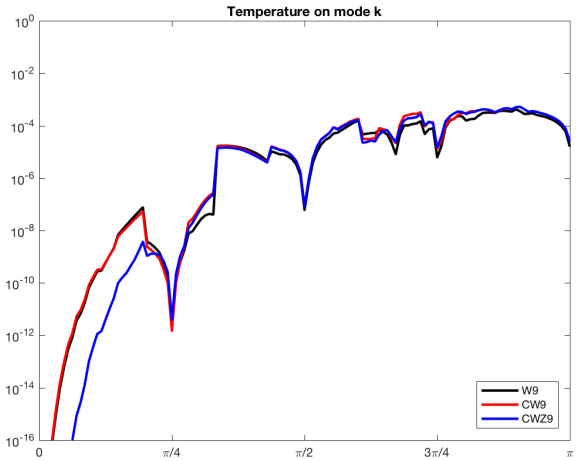


Order 9



- CWENOZ are the coolest schemes retaining non oscillatory

Non oscillatory

A scheme for conservation laws cannot be cold (I mean, with zero temperature), because it would be oscillatory. Some distortion is necessary to prevent spurious oscillations. In this sense, CWENO schemes are cool.

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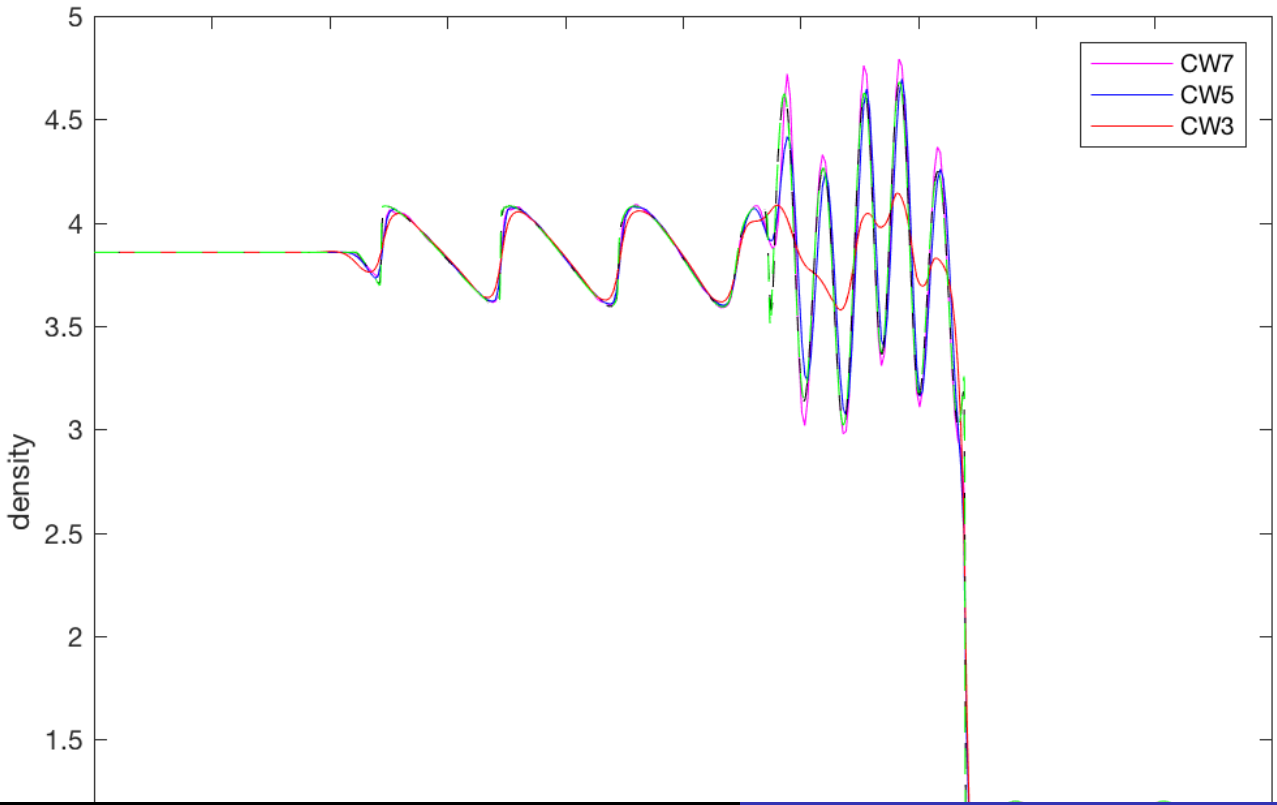


Background


LeVeque Lectures in Math., Birkhäuser, (1992).


Trefethen SIAM Review (1982)

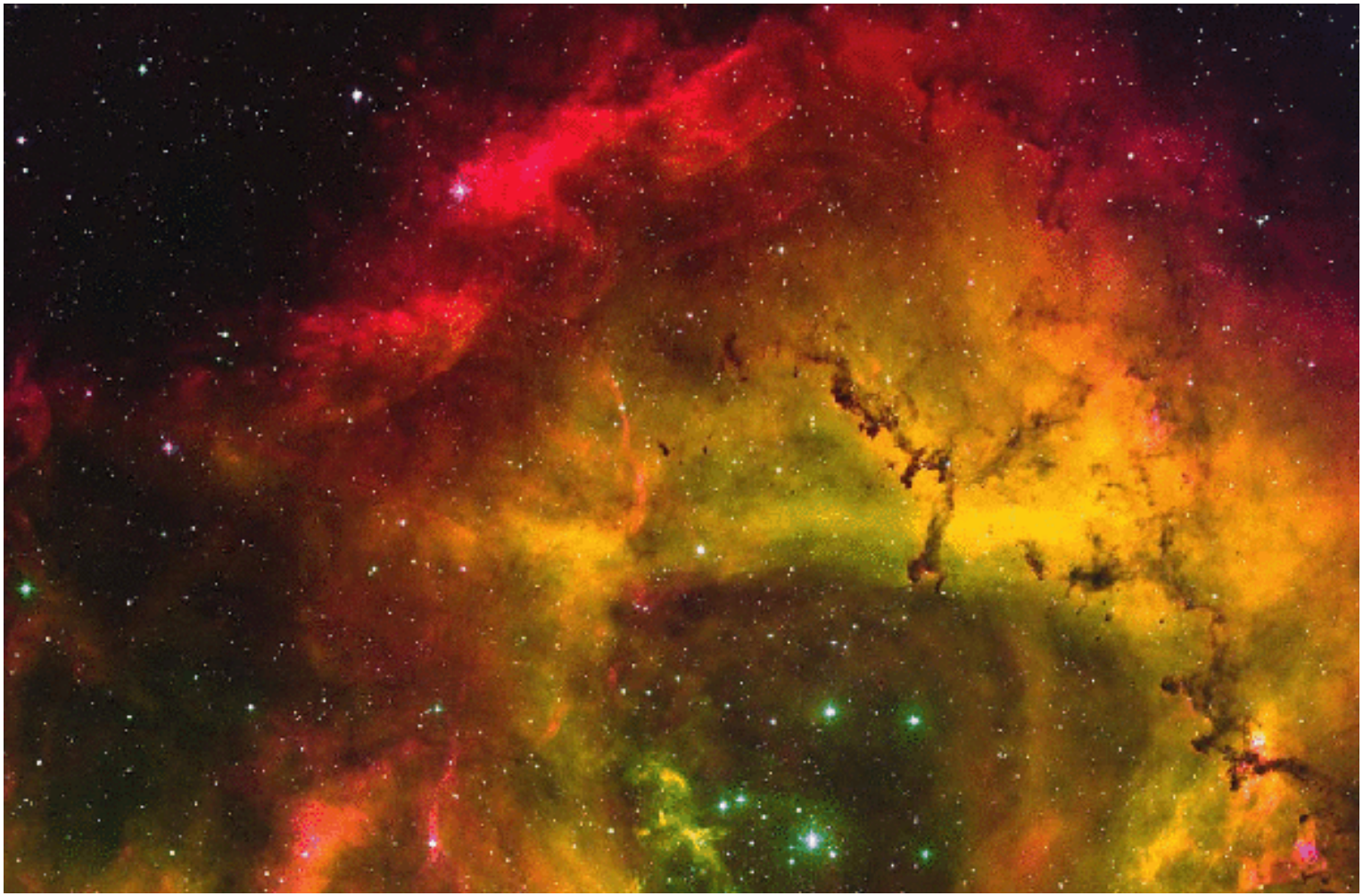

Pirozzoli JCP (2006)


Castro, Costa, Don, JCP. (2011)


Gao, Don, J. Sci. Comp. (2015)


Cravero, P., Semplice, Visconti Comp. Fluids (2017)

The Euler equations with gravity source term



The Euler equations with gravity source term

The Euler equations for a gas in a gravity field, with potential $\Phi(x)$ are

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla_x \cdot \mathbf{f}(\mathbf{u}) = \mathbf{s}(\mathbf{u})$$

$$\begin{cases} \partial_t \rho & +\nabla_x \cdot (\rho \mathbf{v}) & = 0 \\ \partial_t \rho \mathbf{v} & +\nabla_x \cdot (\rho \mathbf{v} \otimes \mathbf{v} + p\mathbf{I}) & = -\rho \nabla \Phi \\ \partial_t E & +\nabla_x \cdot ((E+p)\mathbf{v}) & = -\rho \mathbf{v} \cdot \nabla \Phi \end{cases}$$

Equilibrium solutions

These equations have non trivial steady states when

$$\nabla_x \cdot \mathbf{f}(\mathbf{u}) = \mathbf{s}(\mathbf{u})$$

Here we will focus on hydrostatic steady states, which require

$$v = 0, \qquad \nabla p = -\rho \, \nabla \Phi.$$

Given Φ, these equations do not define a steady state uniquely.

Equilibrium states

We assume we are given two function $\alpha(x)$ and $\beta(x)$ s.t.

$$\rho_{\text{eq}}(x) = \rho_0 \alpha(x) \text{ and } p(x)_{\text{eq}} = p_0 \beta(x), \text{ where } \nabla \beta = \alpha \nabla \Phi$$

Examples

Two typical examples.

- Isothermal equilibrium. Suppose $T \equiv T_{eq}$. Then the functions α and β can be chosen as

$$\rho = \alpha(x) = \frac{1}{T_{eq}} e^{-\frac{\phi(x)}{T_{eq}}}, \qquad p = \beta(x) = e^{-\frac{\phi(x)}{RT_{eq}}}.$$

- Polytropic equilibrium. If the gas is polytropic, i.e. $p = C\rho^{\nu}$,

$$\rho = \alpha(x) = \left(1 - \frac{\nu-1}{\nu}\phi(x)\right)^{1/(\nu-1)}, \quad p(x) = \beta(x) = C\alpha(x)^{\nu}$$

  In particular, if $\nu = \gamma$, the equilibrium is isentropic.

Unbalancing due to the reconstruction

Even if the flow is at equilibrium, the numerical solution may fail to remain stationary. Consider a standard FV scheme

$$\overline{U}_j^{n+1} = \overline{U}_j^n - \frac{\Delta t}{\Delta \mathrm{x}} \left(F_{j+1/2} - F_{j-1/2} \right)$$

Unbalancing due to the reconstruction

The numerical flux for the density equation with numerical diffusion Q is

$$F^{\rho}_{j+1/2} = \frac{1}{2}\left((\rho\vec{v})^{+}_{j+1/2} + (\rho\vec{v})^{-}_{j+1/2}\right)\cdot\vec{n} + Q_{j+1/2}\left(\rho^{+}_{j+1/2} - \rho^{-}_{j+1/2}\right)$$

If $\overline{U}^n_j$ are sampled from an hydrostatic steady-state ($\vec{v} = 0$), then

$$F^{\rho}_{j+1/2} = Q_{j+1/2}\left(\rho^{+}_{j+1/2} - \rho^{-}_{j+1/2}\right)$$

so $\overline{U}^{n+1}_j = \overline{U}^n_j$ is only possible if $\rho^{+}_{j+1/2} = \rho^{-}_{j+1/2}$

Well balanced reconstruction

One key ingredient to the construction of well balanced schemes is

Well-balanced reconstruction

We say that a reconstruction is well balanced if, whenever $\overline{U}_j^n$ are sampled from an equilibrium state u_{eq}, then the reconstructed data satisfy

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If the reconstruction is well-balanced, on a steady-state, the numerical flux is exact:

$$F_{j+1/2} = \mathcal{F}\left(u_{\text{eq}}(x_{j+1/2}), u_{\text{eq}}(x_{j+1/2})\right) = f\left(u_{\text{eq}}(x_{j+1/2})\right)$$

Flux-source compatibility

A well balanced reconstruction is not enough to ensure well balancedness.
Consider in fact the momentum equation. Even if a well balanced reconstruction is used, we have

$$\frac{\overline{U}_j^{n+1} - \overline{U}_j^n}{\Delta t} = -\frac{1}{\Delta x}\left(f(u_{eq}(x_{j+1/2}) - f(u_{eq}(x_{j-1/2})\right) + \overline{S_j(U(x))},$$

and one gets $\overline{U}_j^{n+1} = \overline{U}_j^n$ only if the quadrature rule employed for the source term is properly chosen.

Well-balanced quadrature

The quadrature for the source is well balanced if at equilibrium

$$\frac{1}{\Delta x}\left(f(u_{eq}(x_{j+1/2}) - f(u_{eq}(x_{j-1/2})\right) = \overline{S_j(U(x))}$$

Well balanced reconstruction

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- Let $\overline{r}_j = \overline{\rho}_j - \overline{\alpha}_j$. Then compute $r(x) = \mathcal{R}(\overline{r})(x)$. At equilibrium, $\overline{r}_j \equiv 0, \implies r(x) \equiv 0$.

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Note that ρ, v and p are all continuous across interfaces at equilibrium.

Source term discretization

Given α, β s.t. $\alpha(x)\nabla\Phi = \nabla\beta$, the cell average of the source can be written as

$$\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho\nabla\Phi = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\rho}{\alpha}\nabla\beta.$$

The well balanced quadrature then is

$$\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\rho}{\alpha}\nabla\beta \approx \frac{1}{2}\left(\frac{\rho^{+}_{j-1/2}}{\alpha(x_{j-1/2})} + \frac{\rho^{-}_{j+1/2}}{\alpha(x_{j+1/2})}\right)\frac{\beta(x_{j+1/2}) - \beta(x_{j-1/2})}{\Delta x}$$

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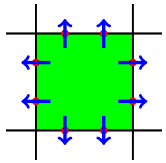
- momentum equation: for any consistent $\mathcal{F}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\rho v}_j = -\frac{\beta(x_{j+1/2}) - \beta(x_{j-1/2})}{\Delta x} + \frac{1}{2}\left(\frac{\alpha(x_{j-1/2})}{\alpha(x_{j-1/2})} + \frac{\alpha(x_{j+1/2})}{\alpha(x_{j+1/2})}\right)\frac{\beta(x_{j+1/2}) - \beta(x_{j-1/2})}{\Delta x}$$

High order source term discretization (2D)

To boost accuracy, use Richiarson extrapolation on the well balanced quadrature[2]

- use Gaussian rules for flux integration on faces:



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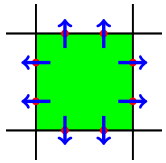
- use Gaussian rules for flux integration on faces: 
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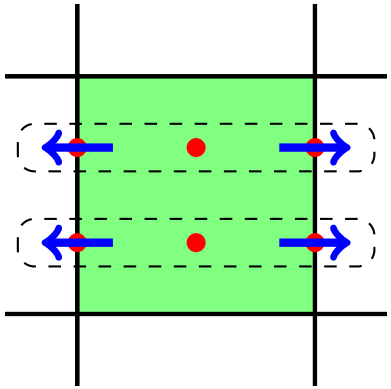

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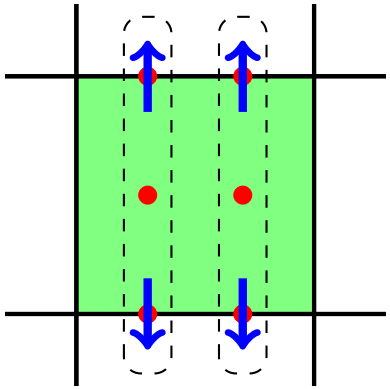

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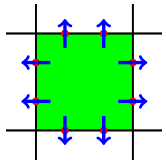
- use Gaussian rules for flux integration on faces: 
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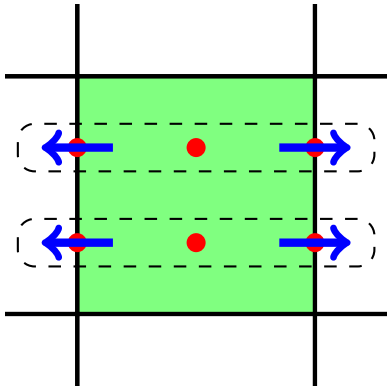
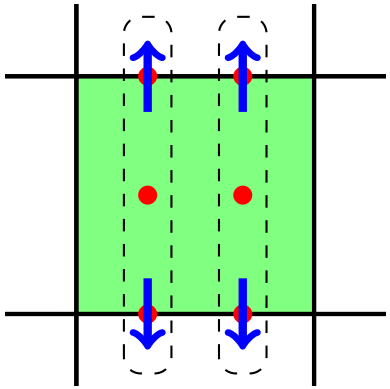


and



- on equilibrium data, the fluxes/source term contribution in each dashed rectangle balance each other exactly

[2] Noelle, Pankratz, P., Natvig, JCP 2006.

Accuracy test in 2D

$$\phi(x,y) = x + y$$
$$\rho_0(t,x,y) = 1 + 0.2\sin(\pi((x+y) - t(u_0+v_0)))$$
$$u(t,x,y) = u_0 = 1 \qquad v(t,x,y) = v_0 = 1$$
$$p(t,x,y) = 4.5 + t(u0+v0) - (x+y) + 0.2\cos(\pi * (x+y-t(u_0+v_0)))/\pi$$

On an $N \times N$ cartesian grid:

	density		energy	
N	error	rate	error	rate
20	6.83e-03		8.91e-03	
40	8.61e-04	2.99	1.16e-03	2.94
80	1.08e-04	3.00	1.46e-04	2.99
160	1.35e-05	3.00	1.82e-05	3.00
320	1.68e-06	3.00	2.27e-06	3.00
640	2.10e-07	3.00	2.84e-07	3.00
1280	2.63e-08	3.00	3.55e-08	3.00

Well-balancing for a general equilibrium

Consider the gravity potential $\phi(x) = -\frac{1}{2}x^2$

Polytropic atmosphere:

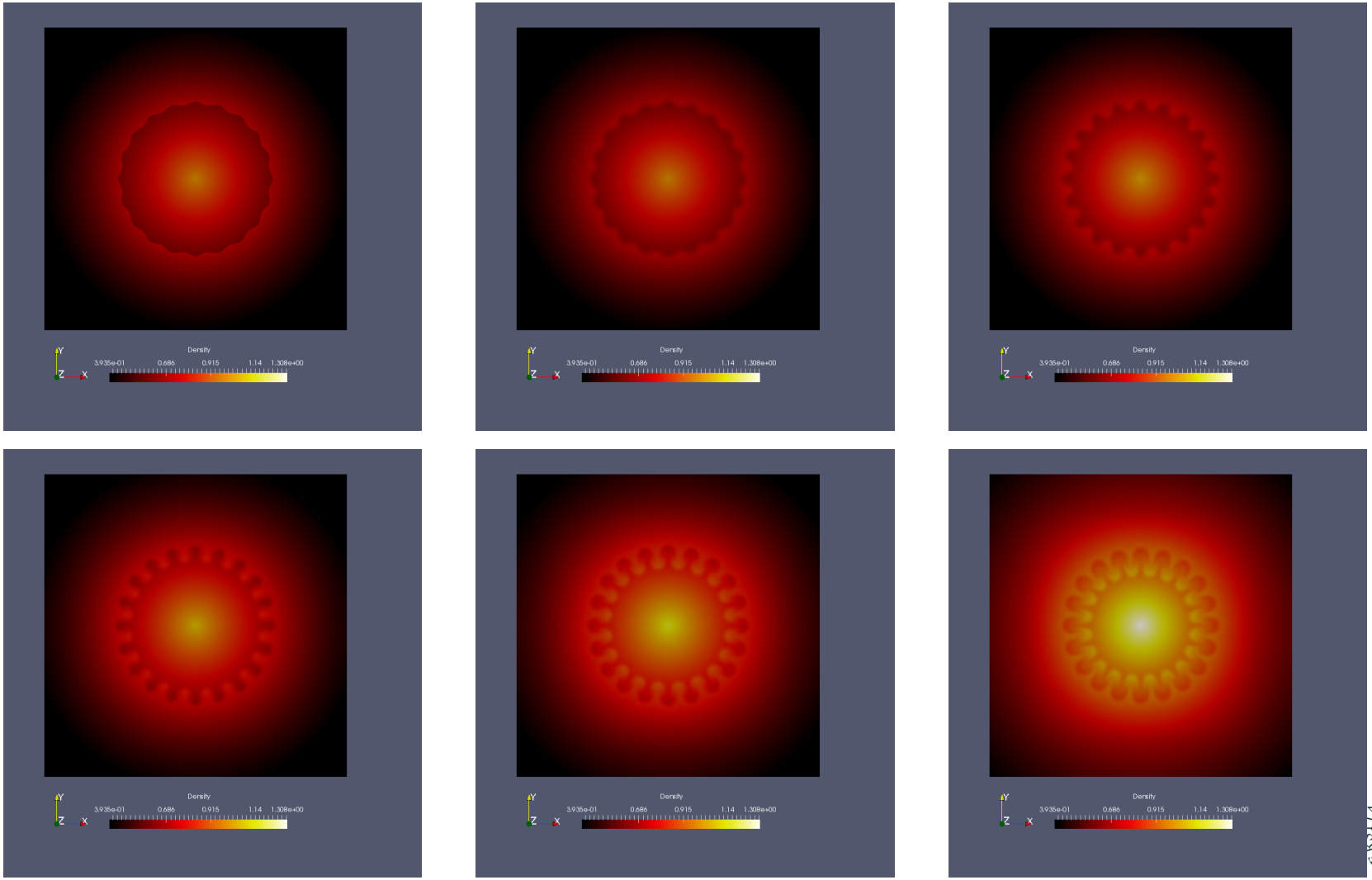
$$\alpha(x) = \left(1 + \frac{\nu-1}{\nu\rho_0}(\phi_0 - \phi(x))\right)^{1/\nu-1}, \quad \beta(x) = \alpha(x)^\nu \qquad \text{(P)}$$

Neither iso-thermal nor polytropic steady-state:

$$\alpha(x) = e^{-x}, \quad \beta(x) = (1+x)\,e^{-x}, \quad T(x) = 1 + x \qquad \text{(G)}$$

	(P)			(G)		
order	ρ	ρu	E	ρ	ρu	E
1	1.62e-16	2.28e-16	4.42e-16	1.16e-17	2.74e-16	1.94e-16
2	2.37e-16	1.60e-16	4.65e-16	1.95e-16	1.93e-16	6.57e-16
3	3.80e-16	2.20e-16	6.77e-16	2.30e-16	1.68e-16	4.76e-16
5	6.05e-16	2.38e-16	8.78e-16	5.22e-16	2.40e-16	8.70e-16

Radial Raylegh-Taylor instability



Background

- Audusse, Bouchut, Bristeau, Klein, Perthame SIAM J. Sci. Comp. (2004)
- Noelle, Pankratz, P., Natvig JCP (2006)
- Chandrashekar, Klingenberg SIAM J. Sci. Comp. (2015)
- Chertock, Cui, Kurganov, Oẍcan, Tadmor. JCP (2018)
- Käppeli, Mishra JCP (2014)
- Li, Xing, JCP (2016)
- Klingenberg, P., Semplice, SIAM J. Sci. Comp., submitted.

Conclusions

- A new paradigm to obtain high order non oscillatory reconstructions has been formalized, which is particularly interesting in the context of balance laws[3].

[3] Cravero, P., Semplice, Visconti, Math. Comp. 2018
[4] Cravero, P., Semplice, Visconti, Comp. Fluids, 2018
[5] Klingenberg, P., Semplice, submitted to SISC

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- We have proposed new well balanced high order schemes for Euler equations with gravity.
- In particular, we can build well balanced schemes also for some moving equilibria, characterized by a constant speed[5].

[3] Cravero, P., Semplice, Visconti, Math. Comp. 2018
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