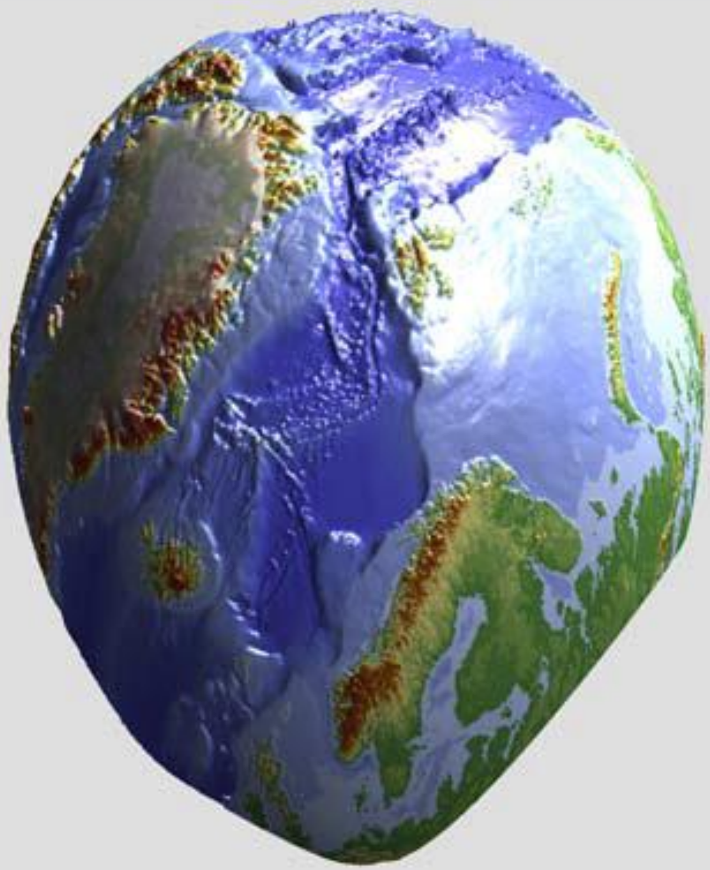
Notes on thin sheet approximation for continental deformations

Sergei Medvedev



Centre for Earth Evolution and Dynamics
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With thanks to:

Yuri Podladchikov & Stefan Schmalholz
(University of Lausanne, Switzerland)
Ebbe Hartz (Aker BP, Norway)
Thibault Duretz (University Rennes, France)

Notes on thin sheet approximation for continental deformations

- England & McKenzie, 1982:
  A thin viscous sheet model for continental deformation (TSA)

- Medvedev & Podladchikov, 1999:
  New extended thin-sheet approximation for geodynamic applications (ETSA)

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Notes on thin sheet approximation for continental deformations

- Thin sheet approximations in geodynamics: TSA and ETSA
- What is thin-sheet approximation?
  - thin lithospheric sheet;
  - thin-sheet equations;
  - thin-sheet approximation
- Characteristic stresses in the lithosphere

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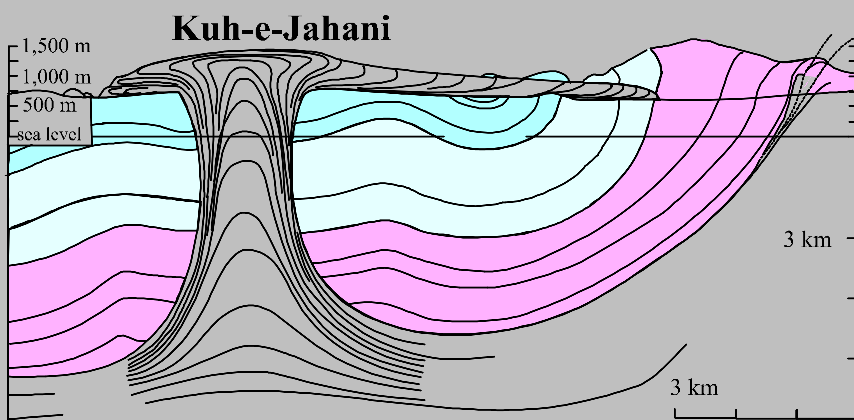
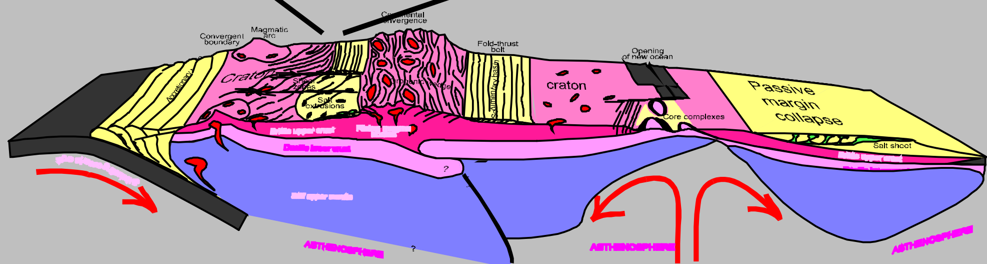
Thin sheets in geodynamics

L~10 km



H~1km

$$\varepsilon = H/L << 1$$



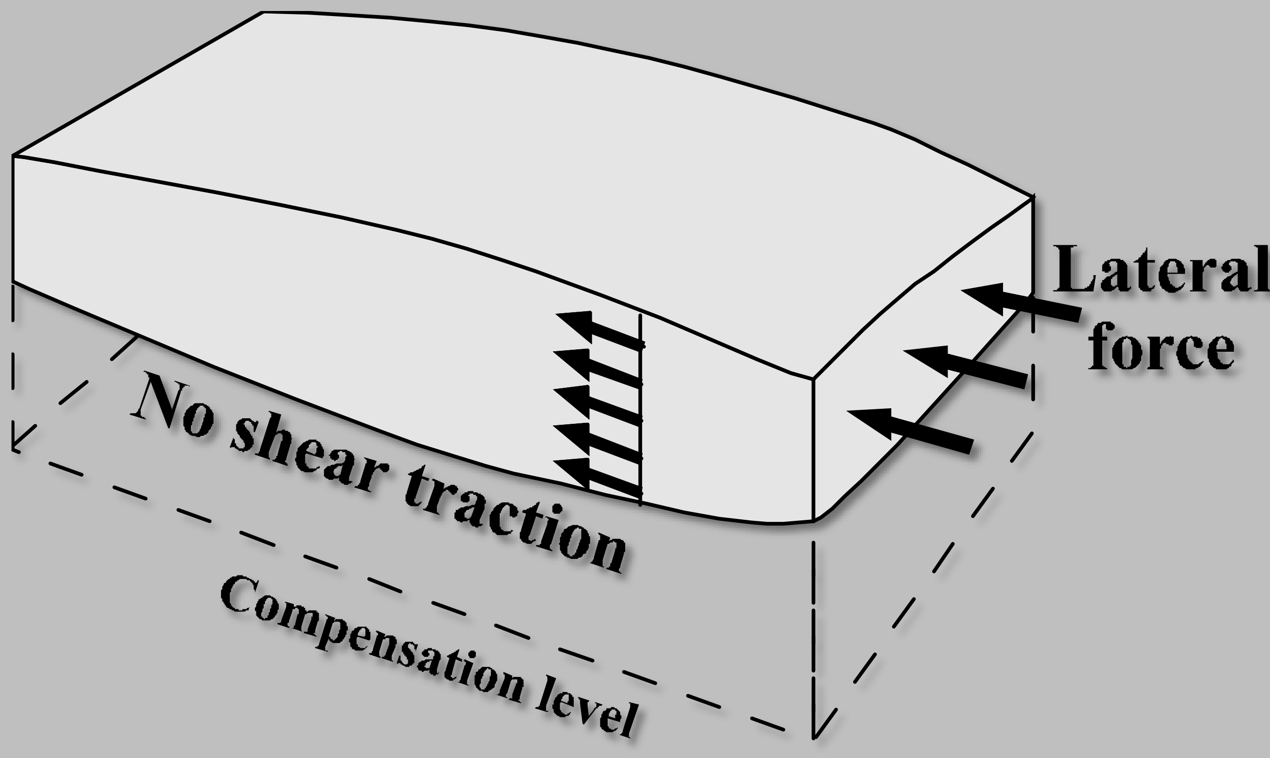
H~100 km

L~1000 km

Existing thin sheet approximations

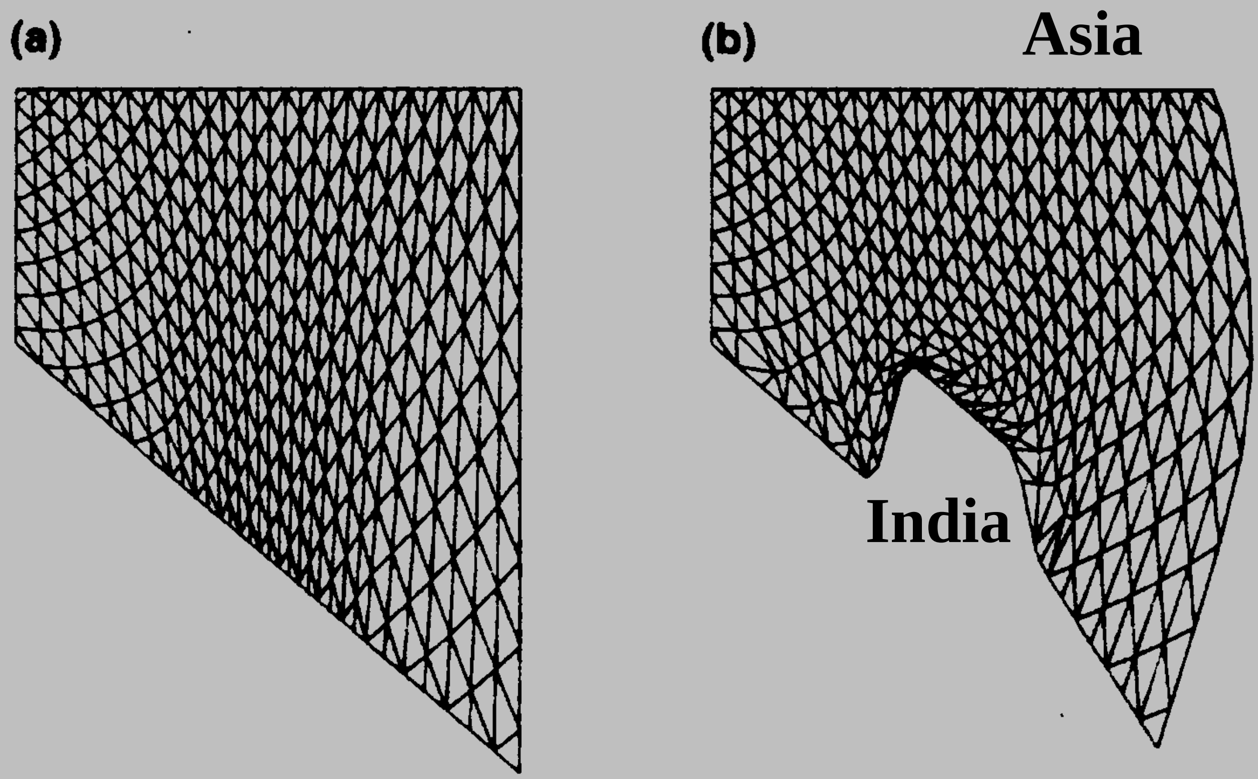
TSA: Constant velocity and no boundary shear (England and McKenzie 1982)

horizontal stresses equilibrate with gravity forces



Existing thin sheet approximations

PS:

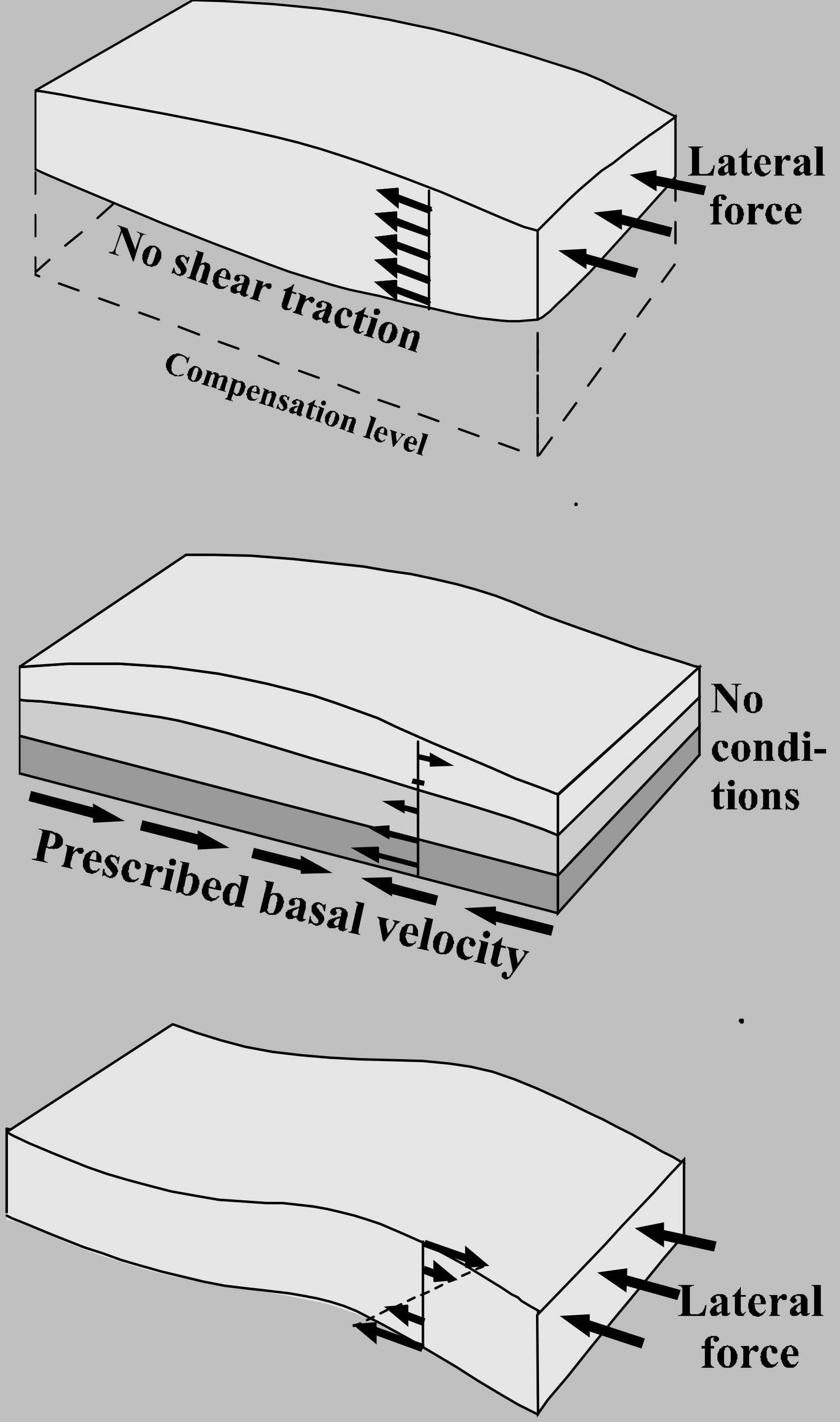


Houseman & England, 1993

Existing thin sheet approximations

Disadvantages:

- Restrictions in boundary conditions
- Restrictions in internal rheological stratification
- Accuracy, oversimplifications



Fundamental rebuilding

Generality:

- Instead of specifications of boundary conditions - relations between internal and external stresses and velocities

Accuracy:

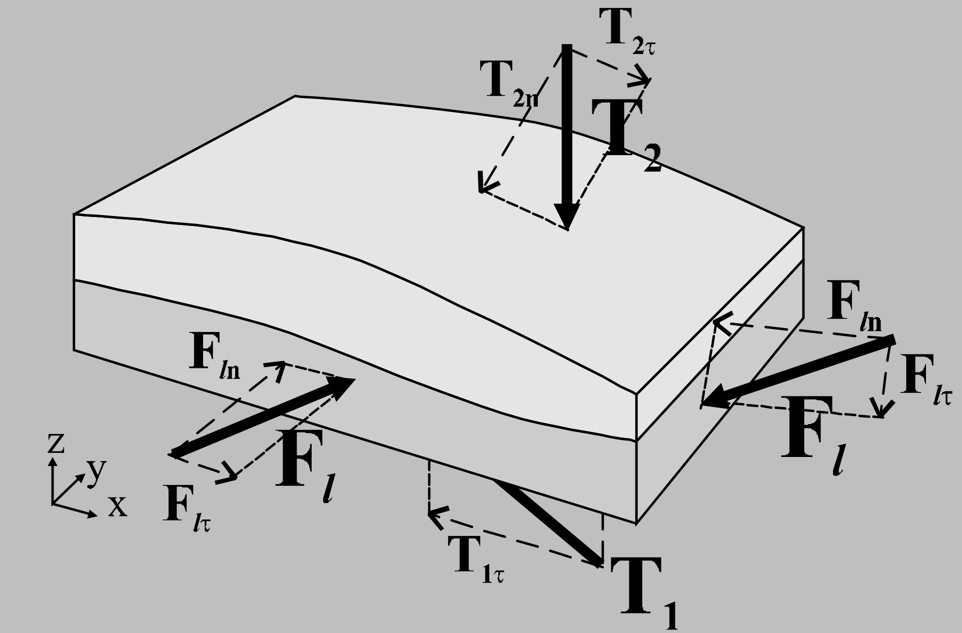
- Increasing the accuracy by keeping more terms in approximations

New approach: ETSA

ETSA

Aims:

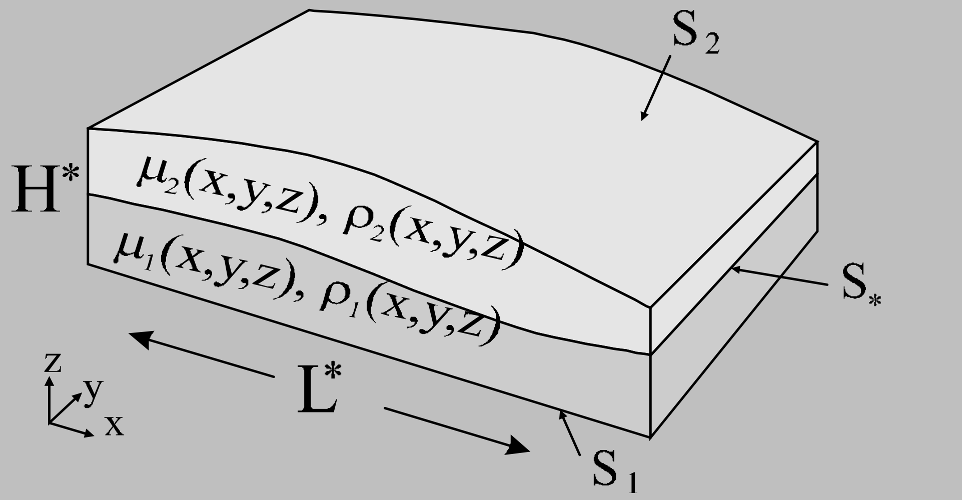
- **Variety of driving forces**
- **Controlled by rheology (assuming large variations of rheology, order of ε)**



Scaling assumption:

- **Small geometric parameter**

$\varepsilon = H^*/L^* << 1$

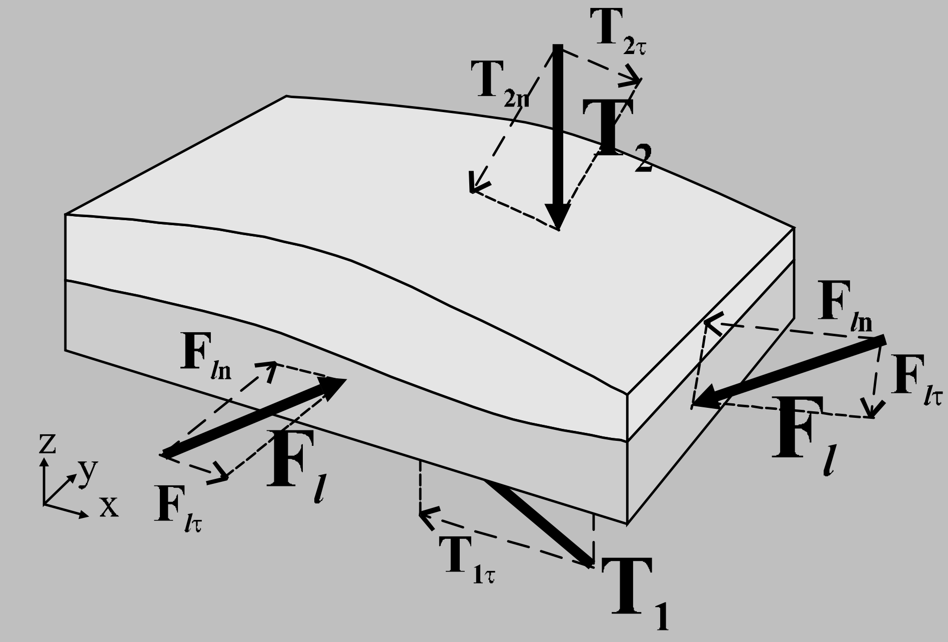


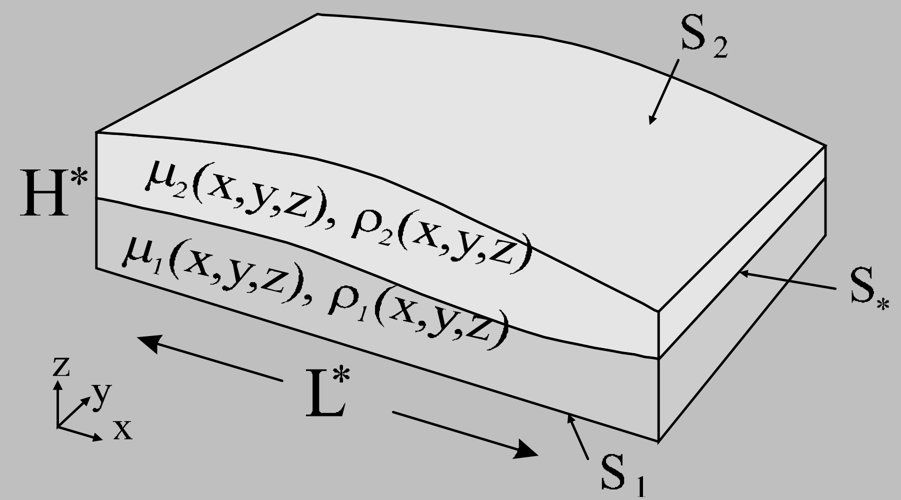
ETSA

Closed system of ETSA:

- **Set of 2D equations of integrated balance of forces and moments in the thin sheet**

- **Rules for reconstruction of 3D stresses and velocities**





$$T_z|_{S_1} = -R_z - \varepsilon^2(a_z - 1)\overline{T'_z} \qquad T_z|_{S_2} = R_z + \overline{\rho} + \varepsilon^2 a_z \overline{T'_z} \ .$$

$$\tau_{ij} = \mu(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}), \quad \tau_{zz} = 2\mu\frac{\partial v_z}{\partial z}, \quad \tau_{iz} = \mu(\varepsilon^2 \cdot \frac{\partial v_z}{\partial x_i} + \frac{\partial v_i}{\partial z})$$

$$\boxed{v_i = V_i + \varepsilon R_i \cdot \int_{S_1}^{z} \tfrac{1}{\mu} dz' - \varepsilon^2 \int_{S_1}^{z} \tfrac{\partial v_z}{\partial x_i} dz' + q_i}$$

$$V_i(x,y) = v_i(x,y,S_1(x,y)) \qquad V_z(x,y) = v_z(x,y,S_1(x,y))$$

$$q_i = \varepsilon^2 \cdot \int_{S_1}^{z} \left(\tfrac{1}{\mu} \int_{S_1}^{z'} \left(\tfrac{\partial P^{(0)}}{\partial x_i} - \tfrac{\partial \tau_{ij}^{(0)}}{\partial x_j} \right) dz'' \right) dz'$$

$$q_i \approx -\varepsilon^2 \cdot \int_{S_1}^{z} \left(\tfrac{1}{\mu} \int_{S_1}^{z'} \tfrac{\partial}{\partial x_i} \left(\int_{S_1}^{z''} \rho dz''' \right) \ dz'' \right) dz' - \varepsilon^2 \cdot \int_{S_1}^{z} \tfrac{(z' - S_1)}{\mu} dz' \cdot \tfrac{\partial R_z}{\partial x_i}$$
$$- \varepsilon^2 \cdot \int_{S_1}^{z} \left(\tfrac{1}{\mu} \int_{S_1}^{z'} \left(\tfrac{\partial}{\partial x_k} (2\mu e_{ik}) + \tfrac{\partial}{\partial x_i} (2\mu e_{kk}) \right) dz'' \right) dz'$$

$$\int_{S_1}^{z} \tfrac{\partial v_z}{\partial x_i} dz' \approx \tfrac{\partial V_z}{\partial x_i}(z - S_1) - \int_{S_1}^{z} \left(\tfrac{\partial}{\partial x_i} \int_{S_1}^{z'} \tilde{e}_{jj} dz'' \right) dz'$$

$$e_{ij} = \tfrac{1}{2}\left(\tfrac{\partial V_i}{\partial x_j} + \tfrac{\partial V_j}{\partial x_i} \right) \qquad \tilde{e}_{ij} = e_{ij} + \tfrac{\varepsilon^2}{2}\left[\tfrac{\partial}{\partial x_j} \left(\int_{S_1}^{z} \tfrac{dz'}{\mu} \cdot R_i \right) + \tfrac{\partial}{\partial x_i} \left(\int_{S_1}^{z} \tfrac{dz'}{\mu} \cdot R_j \right) \right] .$$

$$\tau_{ij} = \mu\left(\tfrac{\partial v_i}{\partial x_j} + \tfrac{\partial v_j}{\partial x_i} \right)$$

$$\boxed{\begin{aligned} \tau_{ij} = \ & 2\mu e_{ij} - 2J_* \tfrac{\partial^2 V_z}{\partial x_i \partial x_j} + J_j \tfrac{\partial V_z}{\partial x_i} + J_i \tfrac{\partial V_z}{\partial x_j} \\ & -2G_{jk} e_{ik} - 2G_{*k} \tfrac{\partial e_{ik}}{\partial x_j} - 2G_{j*} \tfrac{\partial e_{ik}}{\partial x_k} - 2G_{**} \tfrac{\partial^2 e_{ik}}{\partial x_j \partial x_k} \\ & -2G_{ik} e_{jk} - 2G_{*k} \tfrac{\partial e_{jk}}{\partial x_i} - 2G_{i*} \tfrac{\partial e_{jk}}{\partial x_k} - 2G_{**} \tfrac{\partial^2 e_{jk}}{\partial x_i \partial x_k} \\ & +2(F_{ij} - G_{ji} - G_{ji}) e_{kk} + 2(F_{i*} - G_{*i} - G_{i*}) \tfrac{\partial e_{kk}}{\partial x_j} \\ & +2(F_{j*} - G_{*j} - G_{j*}) \tfrac{\partial e_{kk}}{\partial x_i} + 2(F_{**} - 2G_{**}) \tfrac{\partial^2 e_{kk}}{\partial x_i \partial x_j} \\ & +D_* \left(\tfrac{\partial R_i}{\partial x_j} + \tfrac{\partial R_j}{\partial x_i} \right) + D_i R_j + D_j R_i \\ & -2\tilde{D}_{***} \tfrac{\partial^3 R_k}{\partial x_i \partial x_j \partial x_k} - 2\tilde{D}_{i**} \tfrac{\partial^2 R_k}{\partial x_j \partial x_k} - 2\tilde{D}_{j**} \tfrac{\partial^2 R_k}{\partial x_i \partial x_k} - 2\tilde{D}_{**k} \tfrac{\partial^2 R_k}{\partial x_i \partial x_j} \\ & -2\tilde{D}_{i*k} \tfrac{\partial R_k}{\partial x_j} - 2\tilde{D}_{j*k} \tfrac{\partial R_k}{\partial x_i} - (\tilde{D}_{ij*} + \tilde{D}_{ji*}) \tfrac{\partial R_k}{\partial x_k} - (\tilde{D}_{ijk} + \tilde{D}_{jik}) R_k \\ & -2E_* \tfrac{\partial^2 R_z}{\partial x_i \partial x_j} - E_i \tfrac{\partial R_z}{\partial x_j} - E_j \tfrac{\partial R_z}{\partial x_i} - Q_{ij} - Q_{ji} \end{aligned}}$$

$$T_z|_{S_1} = -R_z - \varepsilon^2(a_z - 1)\overline{T'_z} \qquad T_z|_{S_2} = R_z + \overline{\rho} + \varepsilon^2 a_z \overline{T'_z}\ .$$

$$\tau_{ij} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad \tau_{zz} = 2\mu\frac{\partial v_z}{\partial z}, \quad \tau_{iz} = \mu\left(\varepsilon^2 \cdot \frac{\partial v_z}{\partial x_i} + \frac{\partial v_i}{\partial z}\right)$$

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$$V_i(x,y) = v_i(x,y,S_1(x,y)) \qquad V_z(x,y) = v_z(x,y,S_1(x,y))$$

$$q_i = \varepsilon^2 \cdot \int_{S_1}^{z} \left(\frac{1}{\mu}\int_{S_1}^{z'} \left(\frac{\partial P^{(0)}}{\partial x_i} - \frac{\partial \tau_{ij}^{(0)}}{\partial x_j}\right) dz''\right) dz'$$

$$q_i \approx -\varepsilon^2 \cdot \int_{S_1}^{z} \left(\frac{1}{\mu}\int_{S_1}^{z'} \frac{\partial}{\partial x_i}\left(\int_{S_1}^{z''} \rho dz'''\right)\ dz''\right) dz' - \varepsilon^2 \cdot \int_{S_1}^{z} \frac{(z'-S_1)}{\mu} dz' \cdot \frac{\partial R_z}{\partial x_i}$$

$$- \varepsilon^2 \cdot \int_{S_1}^{z} \left(\frac{1}{\mu}\int_{S_1}^{z'} \left(\frac{\partial}{\partial x_k}(2\mu e_{ik}) + \frac{\partial}{\partial x_i}(2\mu e_{kk})\right) dz''\right) dz'$$

Is it worth it?

[illegible]

$$e_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \qquad \tilde{e}_{ij} = e_{ij} + \frac{\varepsilon^2}{2}\left[\frac{\partial}{\partial x_j}\left(\int_{S_1}^{z} \frac{dz'}{\mu} \cdot R_i\right) + \frac{\partial}{\partial x_i}\left(\int_{S_1}^{z} \frac{dz'}{\mu} \cdot R_j\right)\right]\ .$$

$$\tau_{ij} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

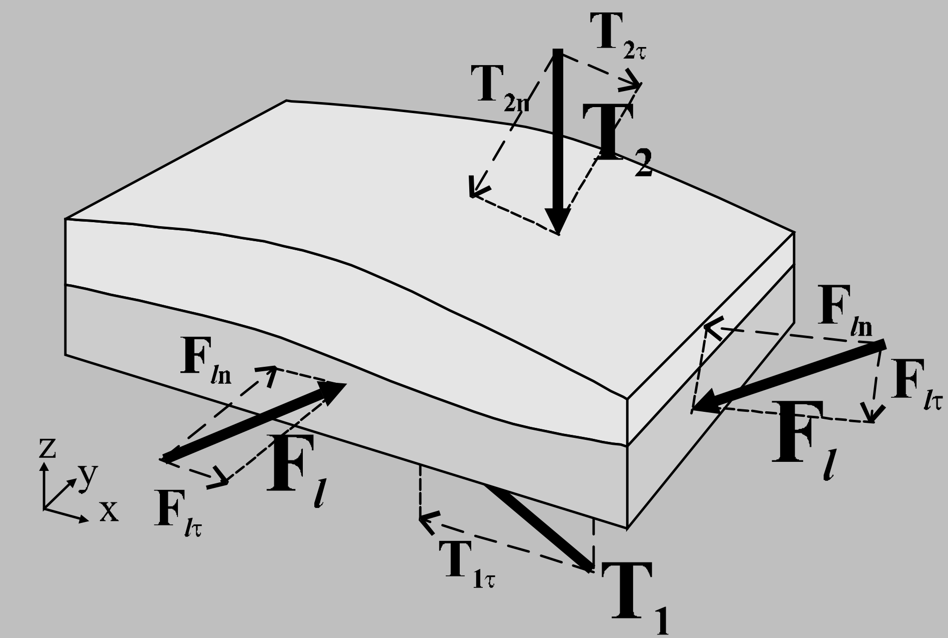
$$\begin{aligned}
\tau_{ij} = \ & 2\mu e_{ij} - 2J_* \frac{\partial^2 V_z}{\partial x_i \partial x_j} + J_j \frac{\partial V_z}{\partial x_i} + J_i \frac{\partial V_z}{\partial x_j} \\
& -2G_{jk} e_{ik} - 2G_{*k} \frac{\partial e_{ik}}{\partial x_j} - 2G_{j*} \frac{\partial e_{ik}}{\partial x_k} - 2G_{**} \frac{\partial^2 e_{ik}}{\partial x_j \partial x_k} \\
& -2G_{ik} e_{jk} - 2G_{*k} \frac{\partial e_{jk}}{\partial x_i} - 2G_{i*} \frac{\partial e_{jk}}{\partial x_k} - 2G_{**} \frac{\partial^2 e_{jk}}{\partial x_i \partial x_k} \\
& +2(F_{ij} - G_{ji} - G_{ji}) e_{kk} + 2(F_{i*} - G_{*i} - G_{i*}) \frac{\partial e_{kk}}{\partial x_j} \\
& +2(F_{j*} - G_{*j} - G_{j*}) \frac{\partial e_{kk}}{\partial x_i} + 2(F_{**} - 2G_{**}) \frac{\partial^2 e_{kk}}{\partial x_i \partial x_j} \\
& +D_* \left(\frac{\partial R_i}{\partial x_j} + \frac{\partial R_j}{\partial x_i}\right) + D_i R_j + D_j R_i \\
& -2\tilde{D}_{***} \frac{\partial^3 R_k}{\partial x_i \partial x_j \partial x_k} - 2\tilde{D}_{i**} \frac{\partial^2 R_k}{\partial x_j \partial x_k} - 2\tilde{D}_{j**} \frac{\partial^2 R_k}{\partial x_i \partial x_k} - 2\tilde{D}_{**k} \frac{\partial^2 R_k}{\partial x_i \partial x_j} \\
& -2\tilde{D}_{i*k} \frac{\partial R_k}{\partial x_j} - 2\tilde{D}_{j*k} \frac{\partial R_k}{\partial x_i} - (\tilde{D}_{ij*} + \tilde{D}_{ji*}) \frac{\partial R_k}{\partial x_k} - (\tilde{D}_{ijk} + \tilde{D}_{jik}) R_k \\
& -2E_* \frac{\partial^2 R_z}{\partial x_i \partial x_j} - E_i \frac{\partial R_z}{\partial x_j} - E_j \frac{\partial R_z}{\partial x_i} - Q_{ij} - Q_{ji}
\end{aligned}$$

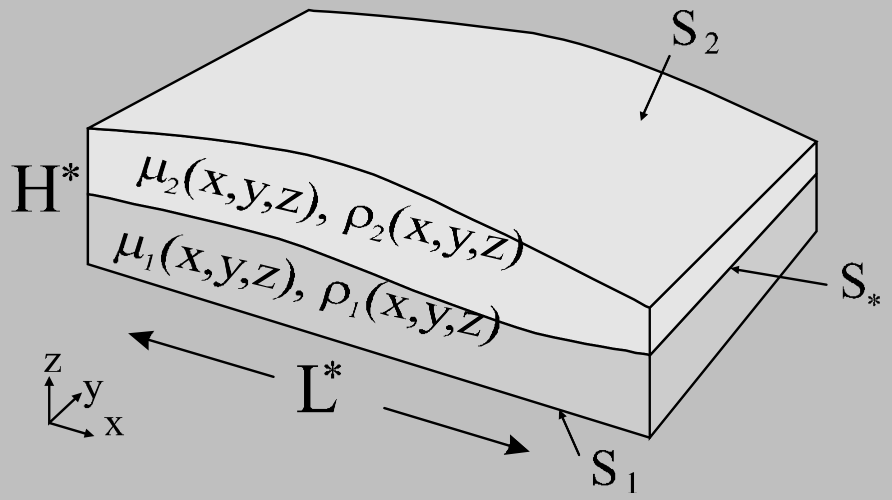
Tests of ETSA

- **Applicability:**

- **Generality:**
  - –Comparison with previous approximations:

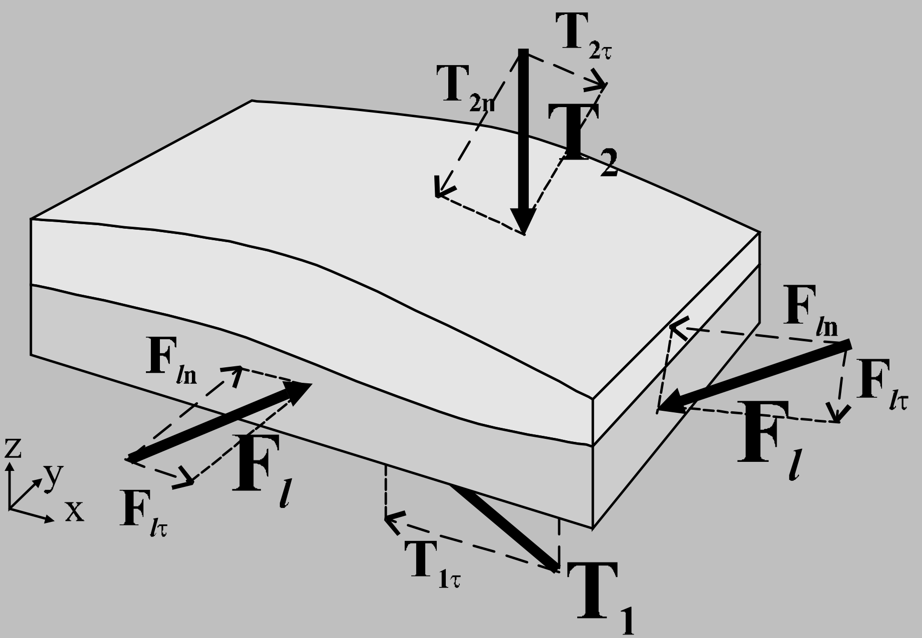
Simplifications + specified boundary conditions = existing approximations

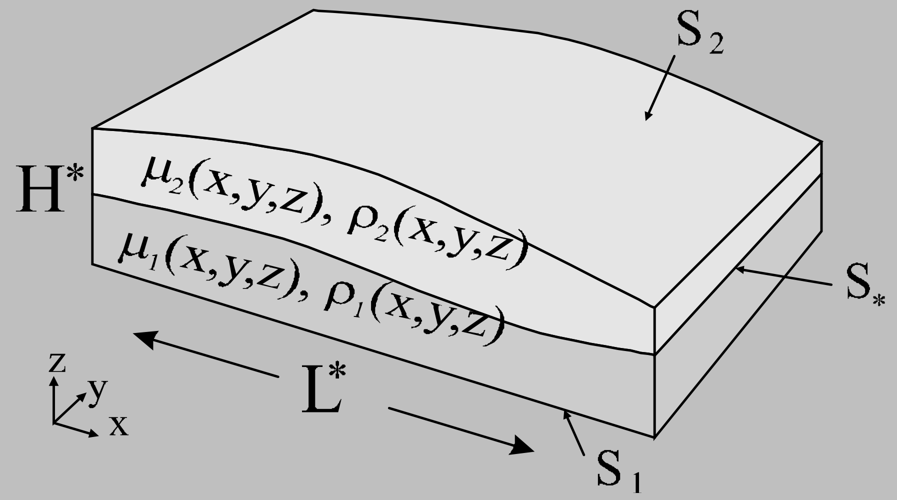




Tests of ETSA

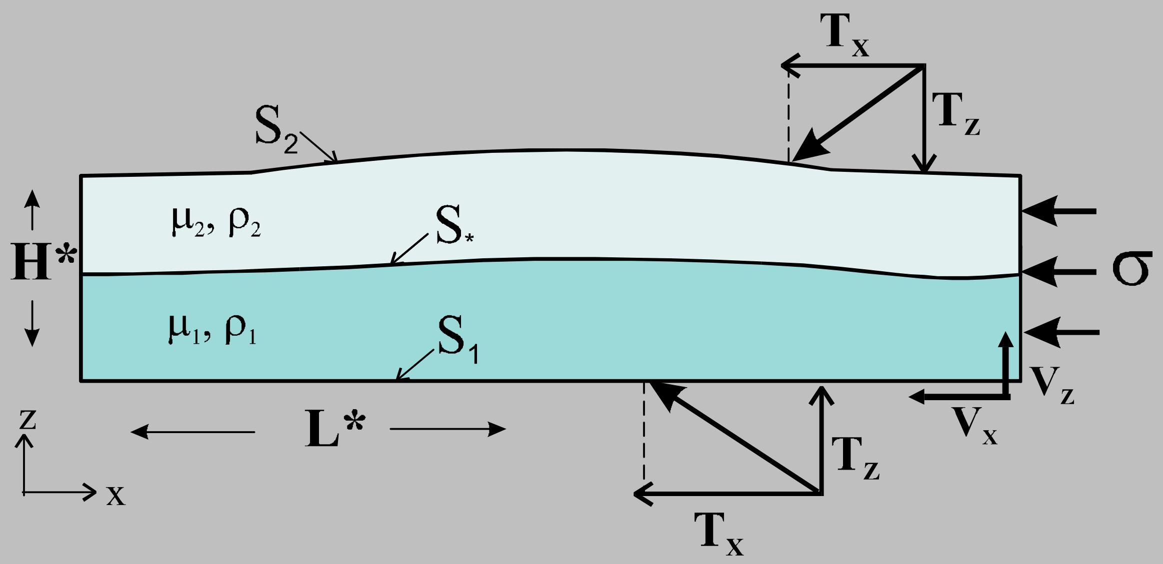
- **Applicability**
- **Generality**
- **2D (cross sectional) analytical tests**





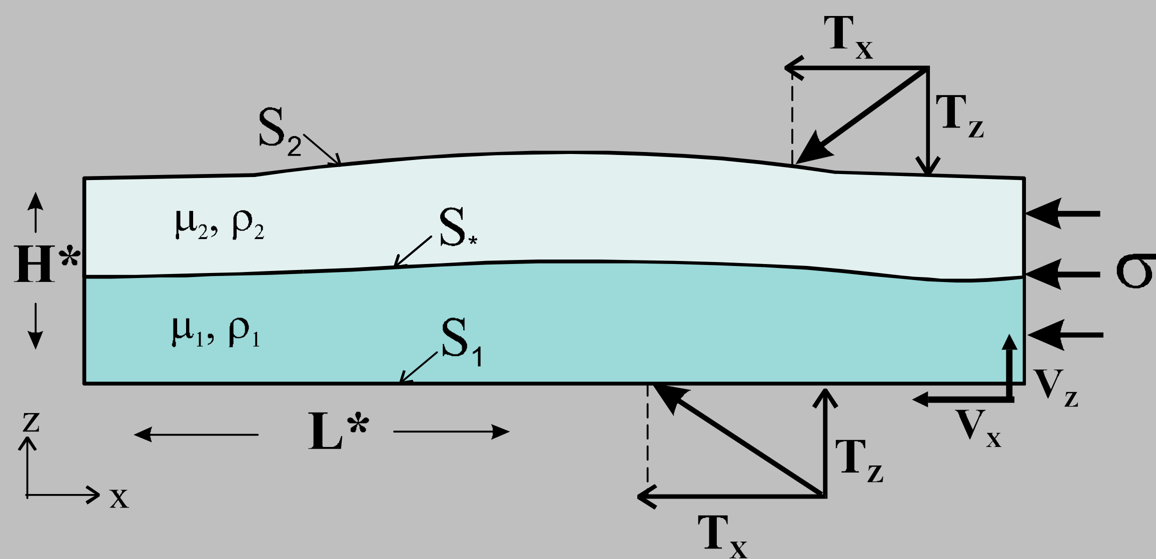
2D tests of ETSA

- **Ability in handling strong competence contrast**
- **Rayleigh-Taylor instability**



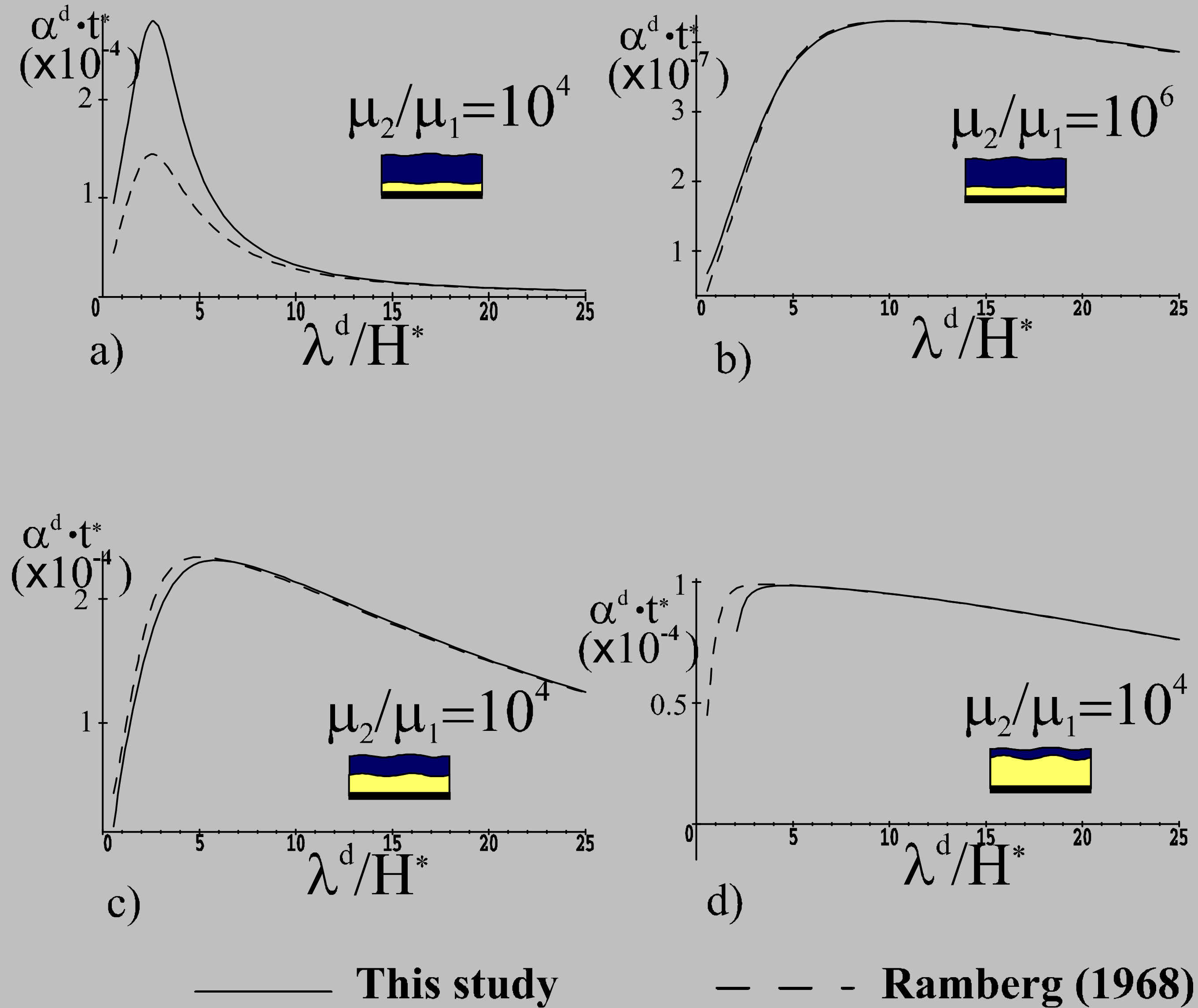
2D tests of ETSA

- **Ability in handling strong competence contrast**
- **Rayleigh-Taylor instability**
- **Small perturbations**
- **Comparison with exact solutions**



2D tests of ETSA, Rayleigh-Taylor instability

spectra



History (fate?) of ETSA

- 10-15 refs of the same type, “The approach we used in our study does not work, most probably we should use ETSA”
- People try to derive some of new equations for advanced thin-sheet approximation. Usual answer, it was already considered in ETSA, but impossible to find…
- Special type of analytical study, where man cannot handle equations, should use computer for analytical derivations
- Help me appreciate simple approaches

Come back to simple! Being as simple as possible

Thin sheet approximation: back to simple

General thin-sheet approximation:

- **Not planned to be used for modelling Rayleigh-Taylor instability**
- **Ability for analytical estimations**
- **Ability for rheology-independent estimations**
- **Simple numerical estimations**

New look at TSA:

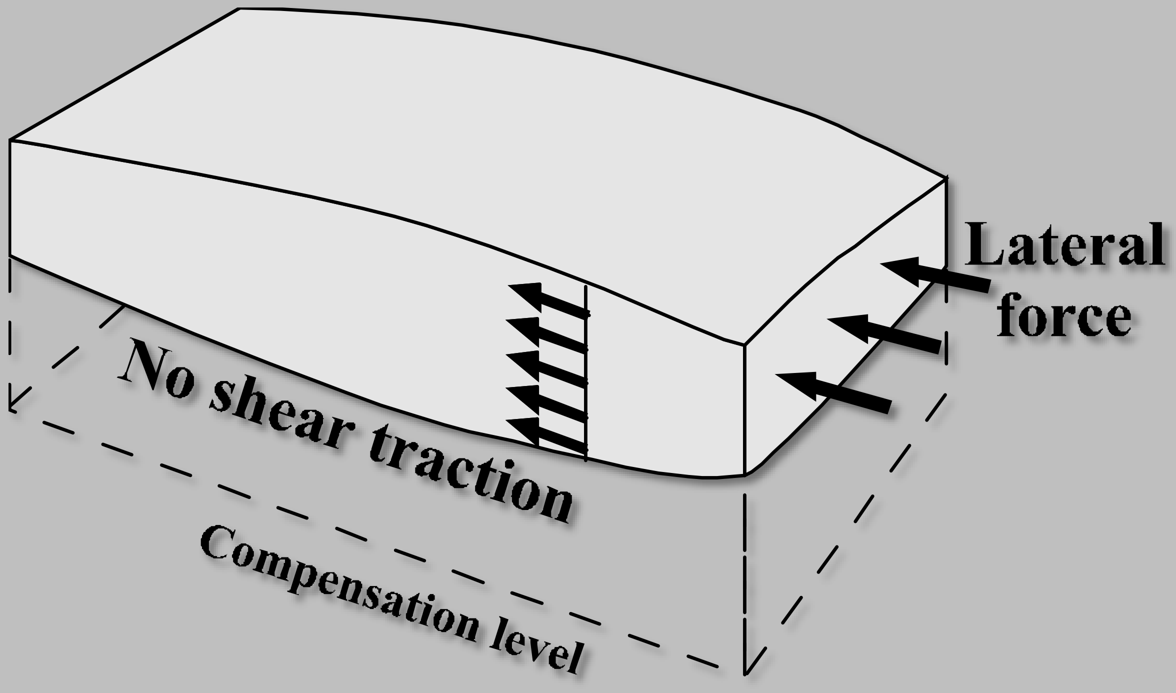
- **What is thin-sheet approximation?**
- **How accurate is it?**
- **ETSA helps: more accurate derivation, not starting from simplifications**

The thin sheet approximation

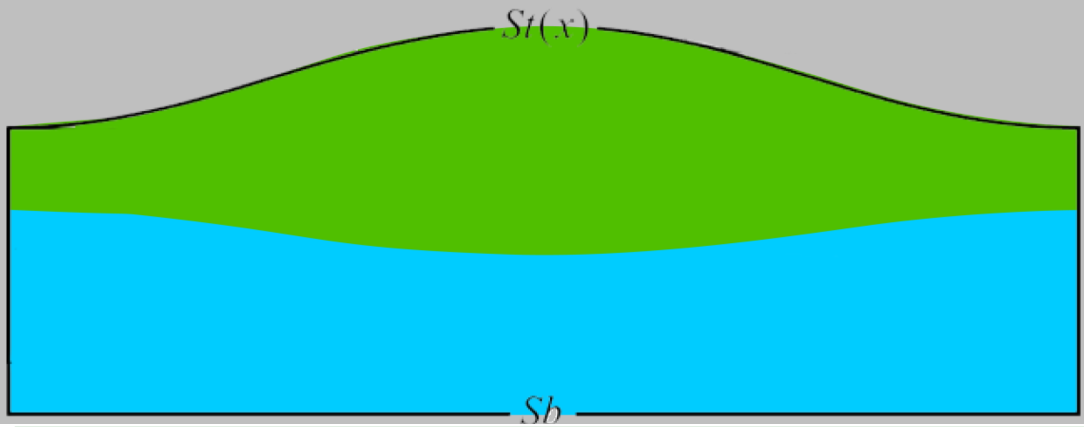
England & McKenzie:

- 1982: A thin viscous sheet model for continental deformation
- 1983: Correction to - a thin viscous sheet model for continental deformation

- Lithosphere scale
- Utilizes weakness of asthenosphere
- Used in many applications
- Overpressure



What is thin sheet approximation?



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Relationship between tectonic overpressure, deviatoric stress, driving force, isostasy and gravitational potential energy

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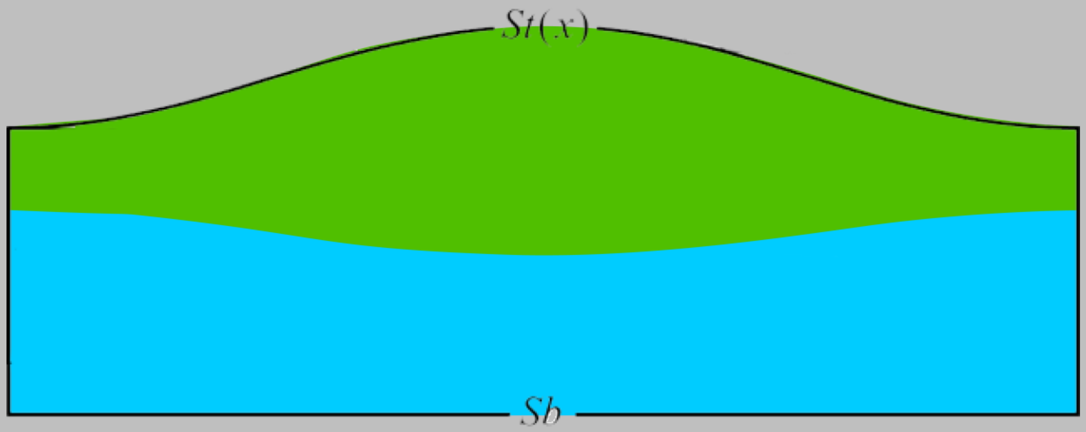
[3] *Geological Institute, ETH Zurich, Zurich, Switzerland*

Schmalholtz et al, 2014

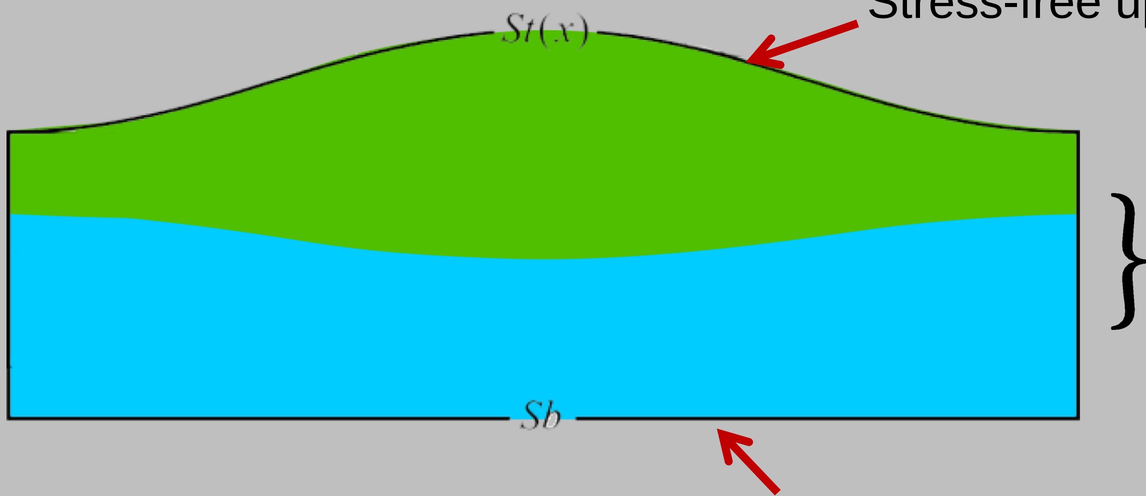
What is thin sheet approximation?



1. What is thin lithospheric sheet?
2. General thin-sheet force balance
3. Lithospheric thin-sheet force balance
4. Thin-sheet approximations

What is thin lithospheric sheet?

Stress-free up



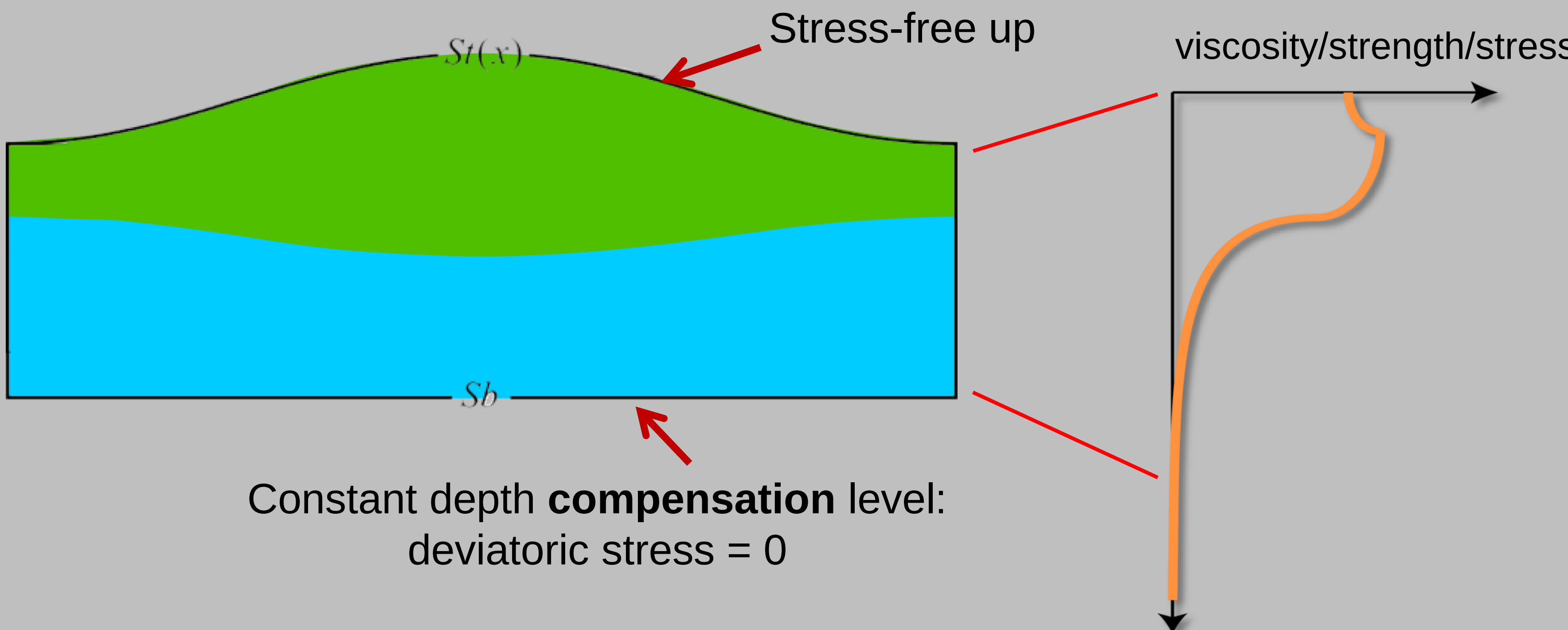
100 – 200 km

Constant depth **compensation** level:
deviatoric stress = 0
(non-material bottom)

1. “Thin-sheet lithosphere” (≠ lithosphere)

- Lithosphere scale
- Utilizes weakness of asthenosphere

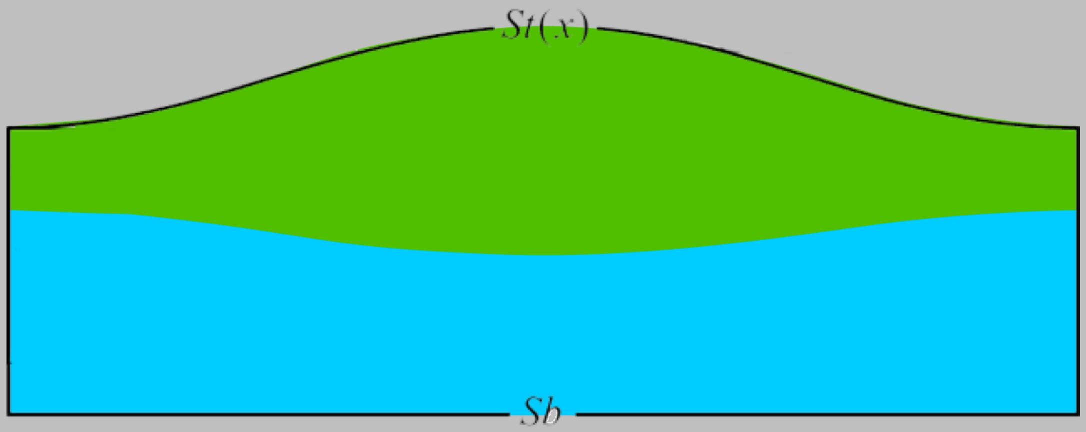
What is thin lithospheric sheet?



1. “Thin-sheet lithosphere” (≠ lithosphere)
2. Thin sheet assumes uneven distribution of strength with depth (≠ averaging)

- Lithosphere scale
- Utilizes weakness of asthenosphere

General thin sheet force balance

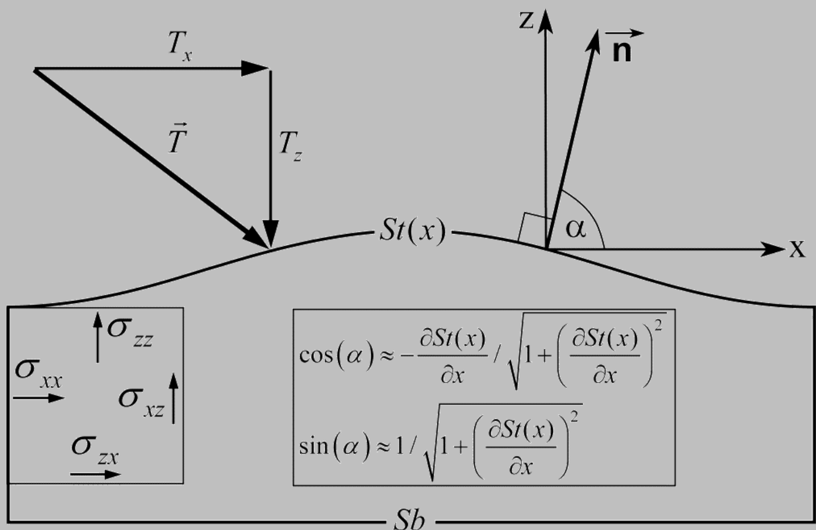


2/3D momentum balance

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = -\rho g$$

General thin sheet force balance



2/3D momentum balance

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

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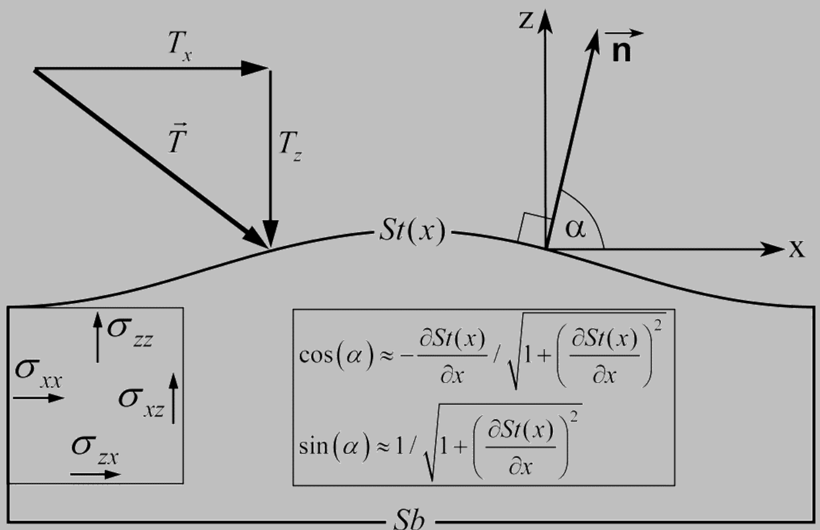
General thin sheet force balance



2/3D momentum balance

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

Depth integration

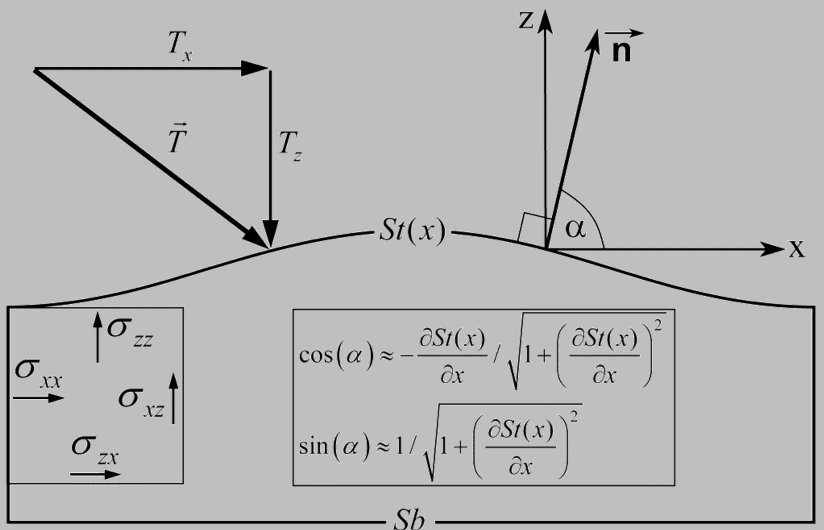
$$\int_{Sb}^{St(x)} \frac{\partial \sigma_{xx}}{\partial x} dz + \int_{Sb}^{St(x)} \frac{\partial \sigma_{xz}}{\partial z} dz = 0$$

$$\int_{Sb}^{St(x)} \frac{\partial \sigma_{xx}}{\partial x} dz + \sigma_{xz}\Big|_{St(x)} - \sigma_{xz}\Big|_{Sb} = 0$$

Change order with differentiation

$$\frac{\partial}{\partial x}\left(\int_{Sb}^{St(x)} \sigma_{xx} dz\right) - \frac{\partial St(x)}{\partial x} \sigma_{xx}\Big|_{St(x)} + \sigma_{xz}\Big|_{St(x)} - \sigma_{xz}\Big|_{Sb} = 0$$

General thin sheet force balance



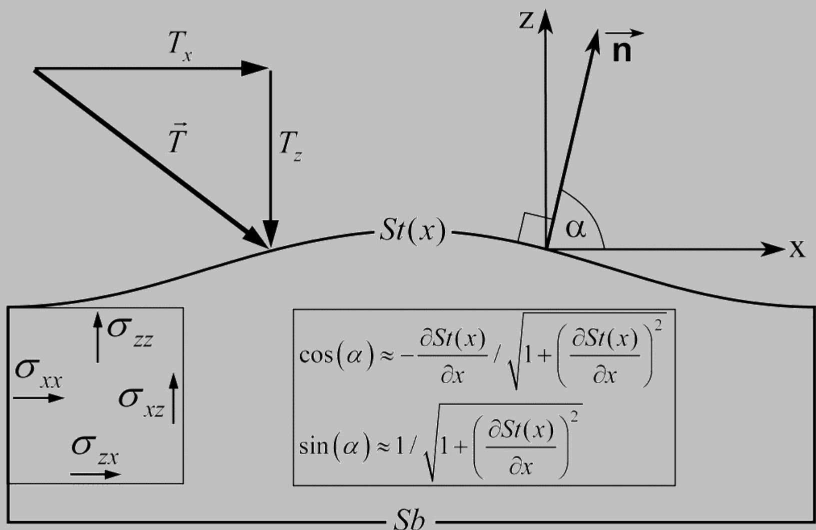
2/3D momentum balance

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

Depth integration

$$\frac{\partial}{\partial x}\left(\int_{Sb}^{St(x)} \sigma_{xx} dz \right) - \frac{\partial St(x)}{\partial x} \sigma_{xx}\Big|_{St(x)} + \sigma_{xz}\Big|_{St(x)} - \sigma_{xz}\Big|_{Sb} = 0$$

General thin sheet force balance



2/3D momentum balance

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

Depth integration

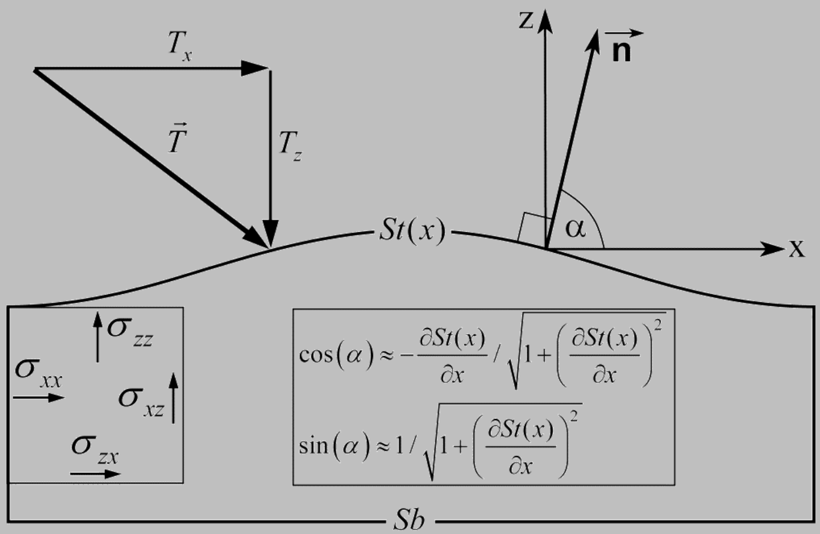
$$\frac{\partial}{\partial x}\left(\int_{Sb}^{St(x)} \sigma_{xx} dz\right) - \frac{\partial St(x)}{\partial x} \sigma_{xx}\Big|_{St(x)} + \sigma_{xz}\Big|_{St(x)} - \sigma_{xz}\Big|_{Sb} = 0$$

$$T_{xt} = T_x\Big|_{St(x)} = \sigma_{xx}\Big|_{St(x)} \cos(\alpha) + \sigma_{xz}\Big|_{St(x)} \sin(\alpha)$$

$$\sqrt{1+\left(\frac{\partial St(x)}{\partial x}\right)^2} \approx 1$$

$$T_{xt} = -\sigma_{xx}\Big|_{St(x)} \frac{\partial St(x)}{\partial x} + \sigma_{xz}\Big|_{St(x)}$$

General thin sheet force balance



2/3D momentum balance

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

Depth integration

$$\frac{\partial}{\partial x}\left(\int_{Sb}^{St(x)} \sigma_{xx} dz\right) - \frac{\partial St(x)}{\partial x}\sigma_{xx}\Big|_{St(x)} + \sigma_{xz}\Big|_{St(x)} - \sigma_{xz}\Big|_{Sb} = 0$$

$$T_{xt} = T_x\Big|_{St(x)} = \sigma_{xx}\Big|_{St(x)} \cos(\alpha) + \sigma_{xz}\Big|_{St(x)} \sin(\alpha)$$

$$\sqrt{1+\left(\frac{\partial St(x)}{\partial x}\right)^2} \approx 1$$

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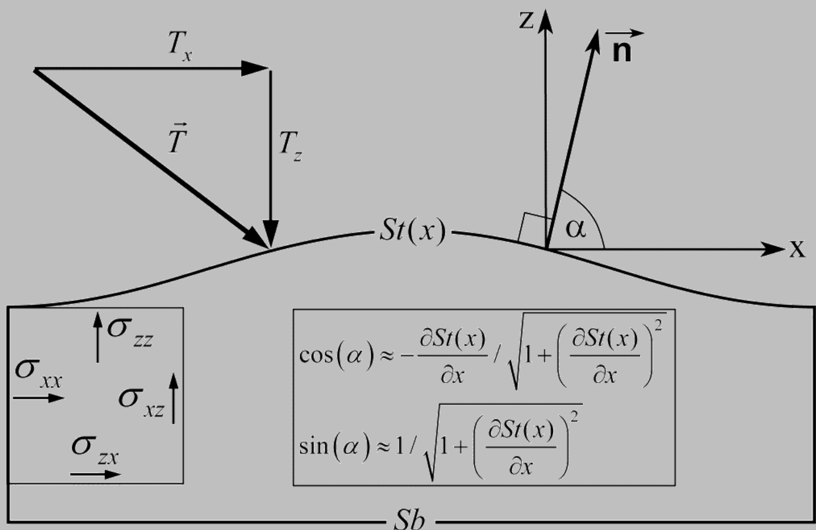
General thin sheet force balance



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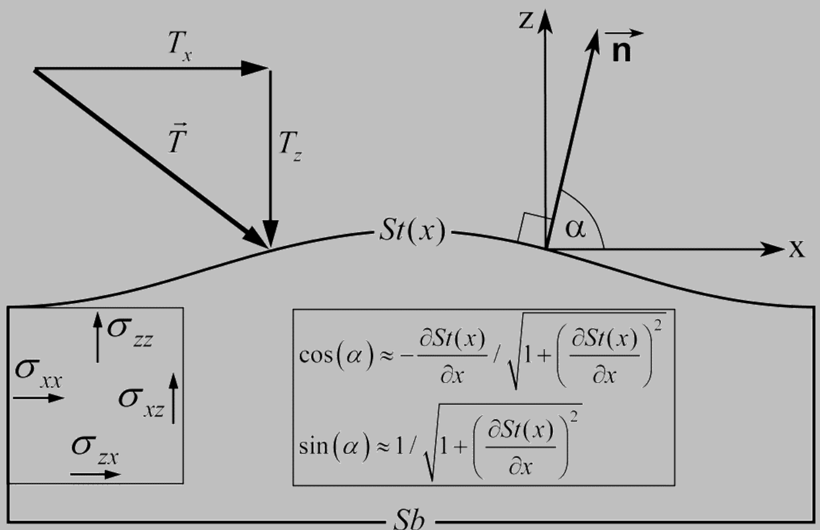
$$\frac{\partial}{\partial x}\left(\int_{Sb}^{St(x)} \sigma_{xx} dz\right) + T_{xt} + T_{xb} = 0$$

General thin sheet force balance



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Depth integration

$$\int_{Sb}^{St(x)} \frac{\partial \sigma_{xx}}{\partial x} dz + \int_{Sb}^{St(x)} \frac{\partial \sigma_{xz}}{\partial z} dz = 0$$

$$\int_{Sb}^{St(x)} \frac{\partial \sigma_{xx}}{\partial x} dz + \sigma_{xz}\big|_{St(x)} - \sigma_{xz}\big|_{Sb} = 0$$

Change order with differentiation

$$\frac{\partial}{\partial x}\left(\int_{Sb}^{St(x)} \sigma_{xx} dz\right) - \frac{\partial St(x)}{\partial x} \sigma_{xx}\big|_{St(x)} + \sigma_{xz}\big|_{St(x)} - \sigma_{xz}\big|_{Sb} = 0$$

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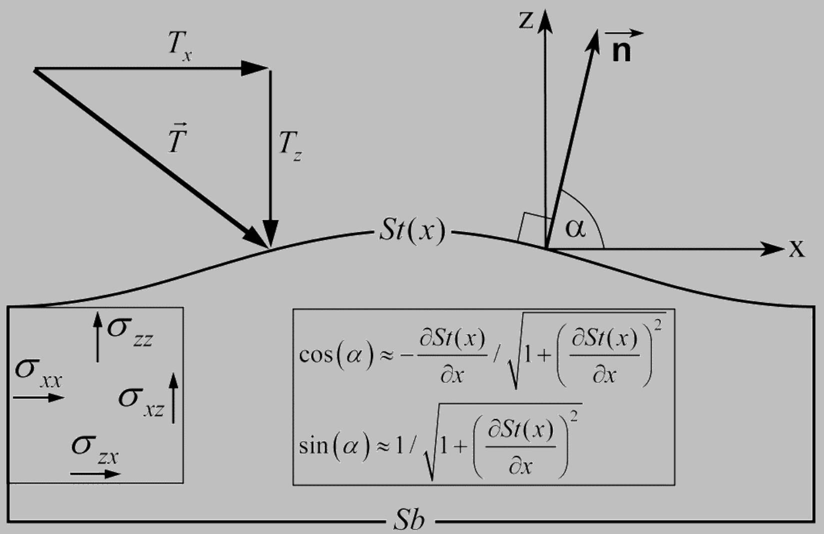

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General thin-sheet equation

$$\frac{\partial}{\partial x}\left(\bar{\sigma}_{xx}\right) + T_{xt} + T_{xb} = 0$$

Change order with differentiation

$$\frac{\partial}{\partial x}\left(\int_{Sb}^{St(x)} \sigma_{xx} dz\right) - \frac{\partial St(x)}{\partial x}\sigma_{xx}\big|_{St(x)} + \sigma_{xz}\big|_{St(x)} - \sigma_{xz}\big|_{Sb} = 0$$

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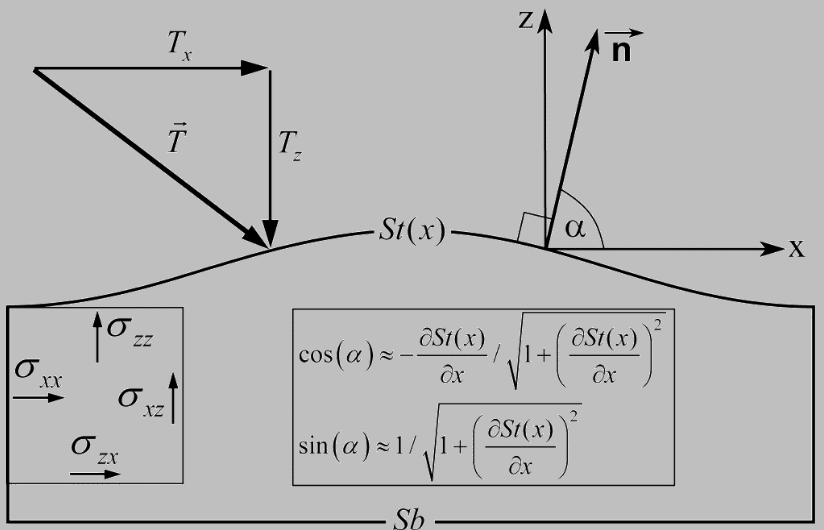
General thin sheet force balance



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Change order with differentiation

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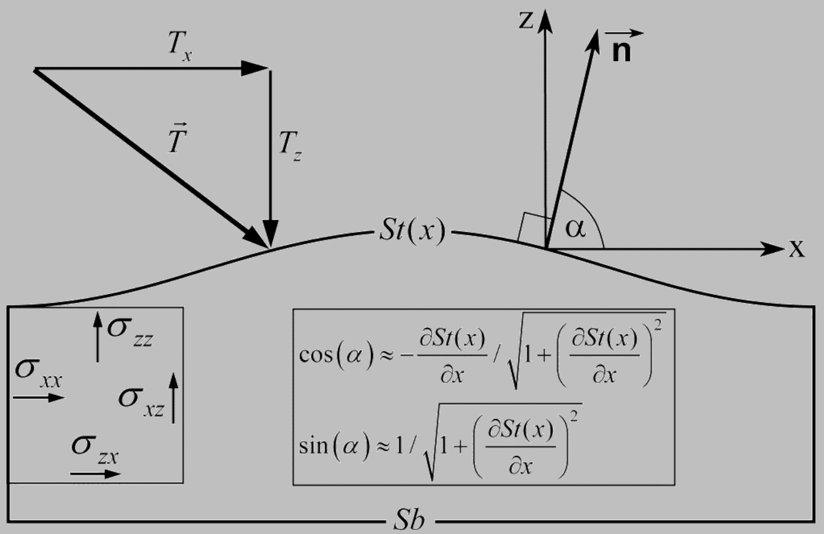
Lithospheric thin sheet force balance



2/3D momentum balance

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

Depth integration

$$\int_{Sb}^{St(x)} \frac{\partial \sigma_{xx}}{\partial x} dz + \int_{Sb}^{St(x)} \frac{\partial \sigma_{xz}}{\partial z} dz = 0$$

$$\int_{Sb}^{St(x)} \frac{\partial \sigma_{xx}}{\partial x} dz + \sigma_{xz}\big|_{St(x)} - \sigma_{xz}\big|_{Sb} = 0$$

Lithospheric thin-sheet equation

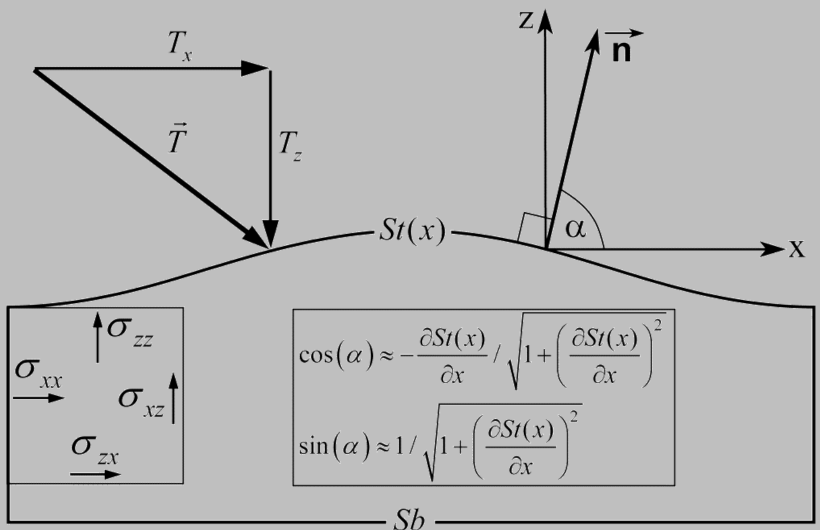
$$\frac{\partial}{\partial x}\left(\bar{\sigma}_{xx}\right) = 0$$

Change order with differentiation

$$\frac{\partial}{\partial x}\left(\int_{Sb}^{St(x)} \sigma_{xx} dz\right) - \frac{\partial St(x)}{\partial x} \sigma_{xx}\big|_{St(x)} + \sigma_{xz}\big|_{St(x)} - \sigma_{xz}\big|_{Sb} = 0$$

$$\frac{\partial}{\partial x}\left(\int_{Sb}^{St(x)} \sigma_{xx} dz\right) + T_{xt} + T_{xb} = 0$$

Lithospheric thin sheet force balance



2/3D momentum balance

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Lithospheric thin-sheet equation

$$\frac{\partial}{\partial x}\left(\bar{\sigma}_{xx}\right) = 0$$

Change order with differentiation

$$\frac{\partial}{\partial x}\left(\int_{Sb}^{St(x)} \sigma_{xx} dz\right) - \frac{\partial St(x)}{\partial x} \sigma_{xx}\big|_{St(x)} + \sigma_{xz}\big|_{St(x)} - \sigma_{xz}\big|_{Sb} = 0$$

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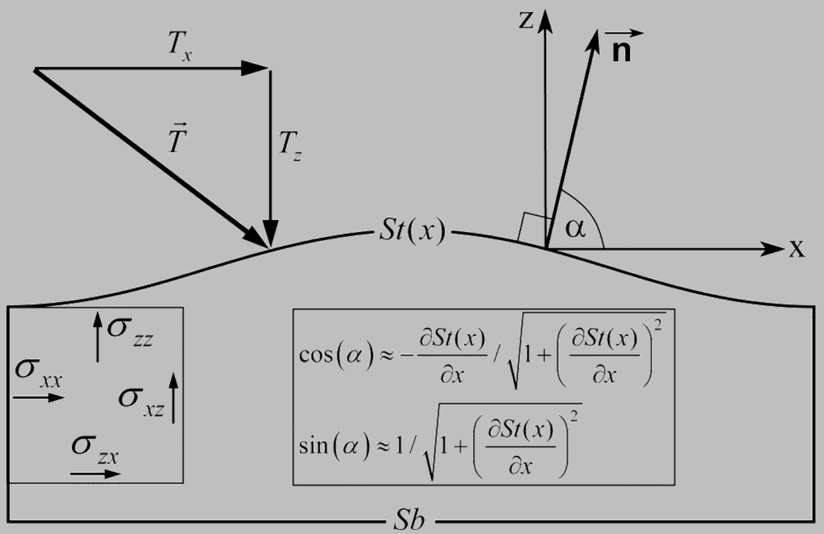
Lithospheric thin sheet force balance



2/3D momentum balance

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

Lithospheric thin-sheet equation

$$\frac{\partial}{\partial x}\left(\bar{\sigma}_{xx}\right) = 0$$

Assumptions (x-proj):

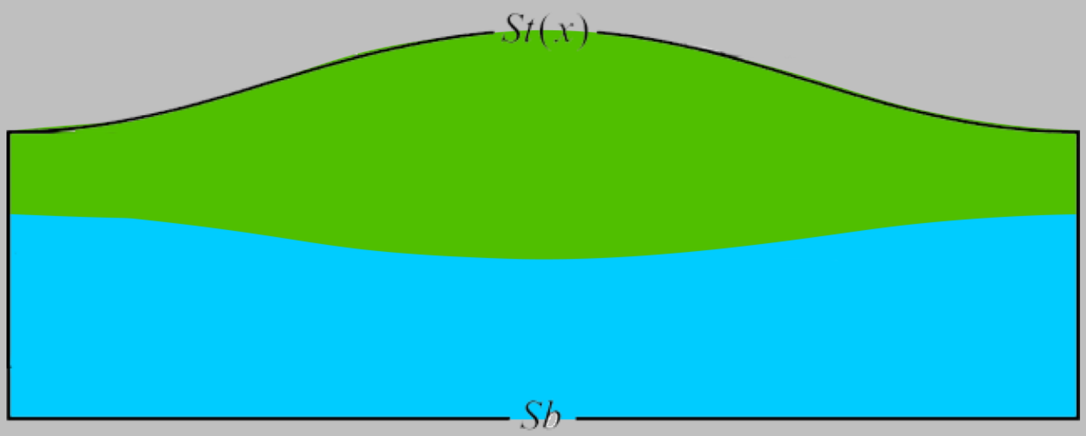
1. $\sqrt{1+\left(\frac{\partial St(x)}{\partial x}\right)^2} \approx 1$

2. Stress-free top

3. Weak base

4. ~~No shear stress~~

Lithospheric thin sheet force balance



2/3D momentum balance

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = -\rho g$$

Lithostatic pressure

$$P_L(x,z) = \int_z^{St(x)} \rho(x,z')\, g\, dz'$$

Shear function

$$Q(x,z) = \frac{\partial}{\partial x} \int_z^{St(x)} \sigma_{xz}\, dz'$$

Lithospheric thin-sheet equations

$$\frac{\partial}{\partial x}\left(\bar{\sigma}_{xx}\right) = 0$$

$$\sigma_{zz}(x,z) = -P_L(x,z) - Q(x,z)$$

Lithospheric thin sheet force balance

Lithospheric thin-sheet equations

$$\frac{\partial}{\partial x}\left(\bar{\sigma}_{xx}\right)=0$$

$$-\sigma_{zz}(x,z)-Q(x,z)=P_L(x,z)$$

$$\frac{\partial}{\partial x}\left(\bar{\sigma}_{xx}-\bar{\sigma}_{zz}-\bar{Q}\right)=\frac{\partial}{\partial x}\left(\bar{P}_L\right)$$

Lithostatic pressure

$$P_L(x,z)=\int_z^{St(x)}\rho(x,z')g\,dz'$$

Shear function

$$Q(x,z)=\frac{\partial}{\partial x}\int_z^{St(x)}\sigma_{xz}\,dz'$$

Lithospheric thin sheet force balance

Lithospheric thin-sheet equations

$$\frac{\partial}{\partial x}\left(\bar{\sigma}_{xx}\right)=0$$

$$-\sigma_{zz}(x,z)-Q(x,z)=P_L(x,z)$$

$$\frac{\partial}{\partial x}\left(\bar{\sigma}_{xx}-\bar{\sigma}_{zz}-\bar{Q}\right)=\frac{\partial}{\partial x}\left(\bar{P}_L\right)$$

$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx}-\bar{Q}\right)=\frac{\partial}{\partial x}\left(GPE\right)$$

Lithostatic pressure

$$P_L\left(x,z\right)=\int_{z}^{St(x)}\rho\left(x,z'\right)g\,dz'$$

Shear function

$$Q(x,z)=\frac{\partial}{\partial x}\int_{z}^{St(x)}\sigma_{xz}\,dz'$$

Gravitational Potential Energy

$$GPE(x)=\int_{Sb}^{St(x)}P_L\left(x,z\right)dz+const$$

Deviatoric stresses:

$$\bar{\sigma}_{ij}=\bar{\tau}_{ij}-\bar{P}\,\delta_{ij}$$

$$\bar{\tau}_{ii}=0$$

Lithospheric thin sheet force balance

Lithospheric thin-sheet equations

$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx} - \bar{Q}\right) = \frac{\partial}{\partial x}\left(GPE\right)$$

$$-\sigma_{zz}(x,z) - Q(x,z) = P_L(x,z)$$

$$\frac{\partial}{\partial x}\left(\bar{\tau}_{xz}\right) = P(x,Sb) - P_L(x,Sb)$$

Lithostatic pressure

$$P_L(x,z) = \int_z^{St(x)} \rho(x,z')\,g\,dz'$$

Shear function

$$Q(x,z) = \frac{\partial}{\partial x}\int_z^{St(x)} \sigma_{xz}\,dz'$$

Gravitational Potential Energy

$$GPE(x) = \int_{Sb}^{St(x)} P_L(x,z)\,dz + const$$

Deviatoric stresses:

$$\bar{\sigma}_{ij} = \bar{\tau}_{ij} - \bar{P}\,\delta_{ij}$$

$$\bar{\tau}_{ii} = 0$$

Lithospheric thin sheet force balance

System of thin-sheet equations

$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx} - \bar{Q}\right) = \frac{\partial}{\partial x}\left(GPE\right)$$

$$\frac{\partial}{\partial x}\left(\bar{\tau}_{xz}\right) = P(x,Sb) - P_L(x,Sb)$$

Lithostatic pressure

$$P_L\left(x,z\right) = \int_{z}^{St(x)} \rho\left(x,z'\right) g dz'$$

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Lithospheric thin sheet force balance

Thin-sheet equations

$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx} - \bar{Q}\right) = \frac{\partial}{\partial x}\left(GPE\right)$$

$$\frac{\partial}{\partial x}\left(\bar{\tau}_{xz}\right) = P(x,Sb) - P_L(x,Sb)$$

Thin-sheet approximation

$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx}\right) = \frac{\partial}{\partial x}\left(GPE\right)$$

$$P(x,Sb) = P_L(x,Sb)$$

Assumptions

1. $\sqrt{1+\left(\frac{\partial St(x)}{\partial x}\right)^2} = 1$

2. Stress-free top

3. Weak base

4ts. $\int_{Sb}^{St(x)}\left(\frac{\partial}{\partial x}\int_{z}^{St(x)}\tau_{xz}dz'\right)dz = const$

5ts. $\bar{\tau}_{xz} = const$

6ts. ~~$\tau_{xz} = 0$~~

Lithospheric thin sheet force balance

Thin-sheet equations

$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx} - \bar{Q}\right) = \frac{\partial}{\partial x}\left(GPE\right)$$

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Thin-sheet approximation

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Testing equations

$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx} - \bar{Q}\right) = \frac{\partial}{\partial x}\left(GPE\right)$$

$$\frac{\partial}{\partial x}\left(\bar{\tau}_{xz}\right) = P(x,Sb) - P_L(x,Sb)$$

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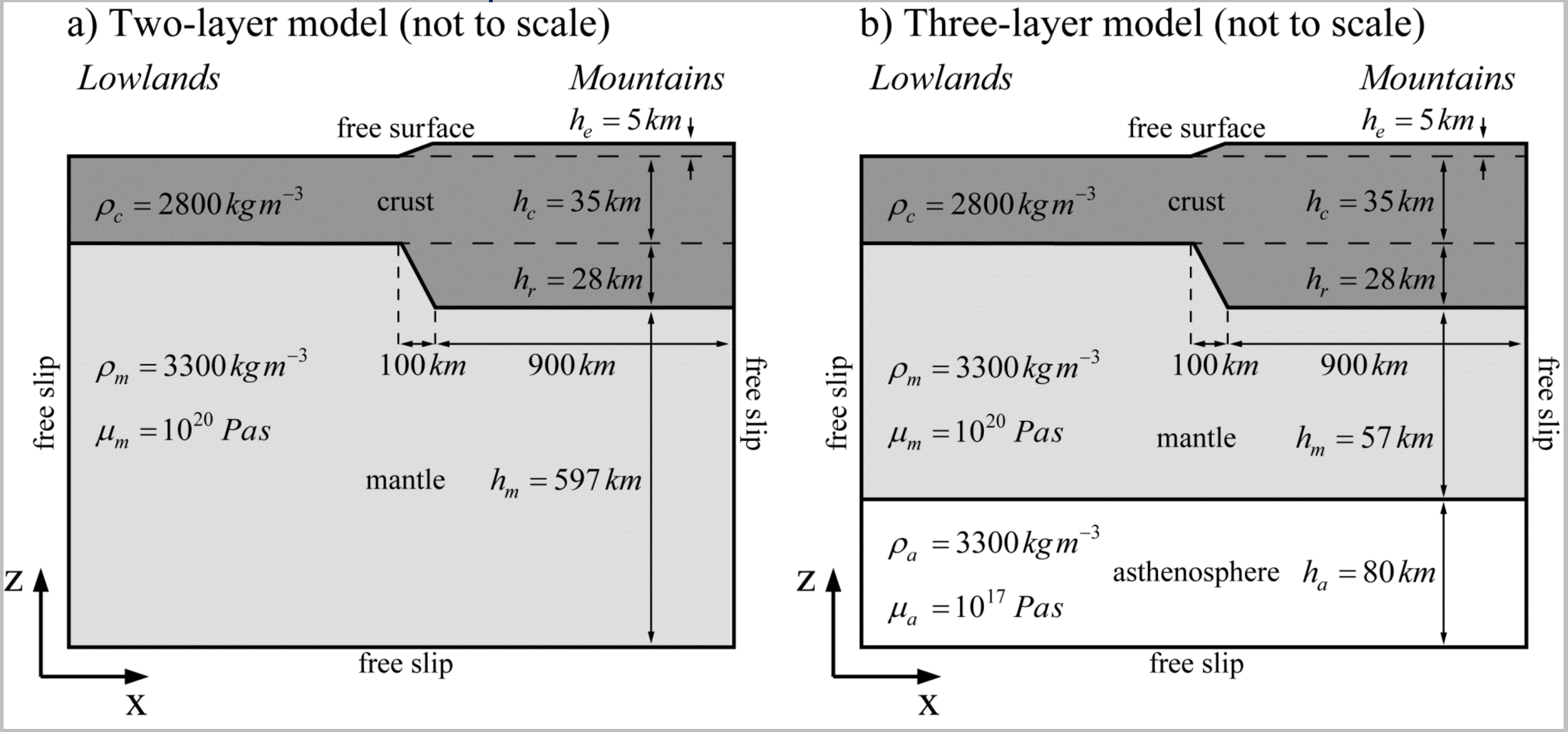

Schmalholtz et al, 2014

Testing equations

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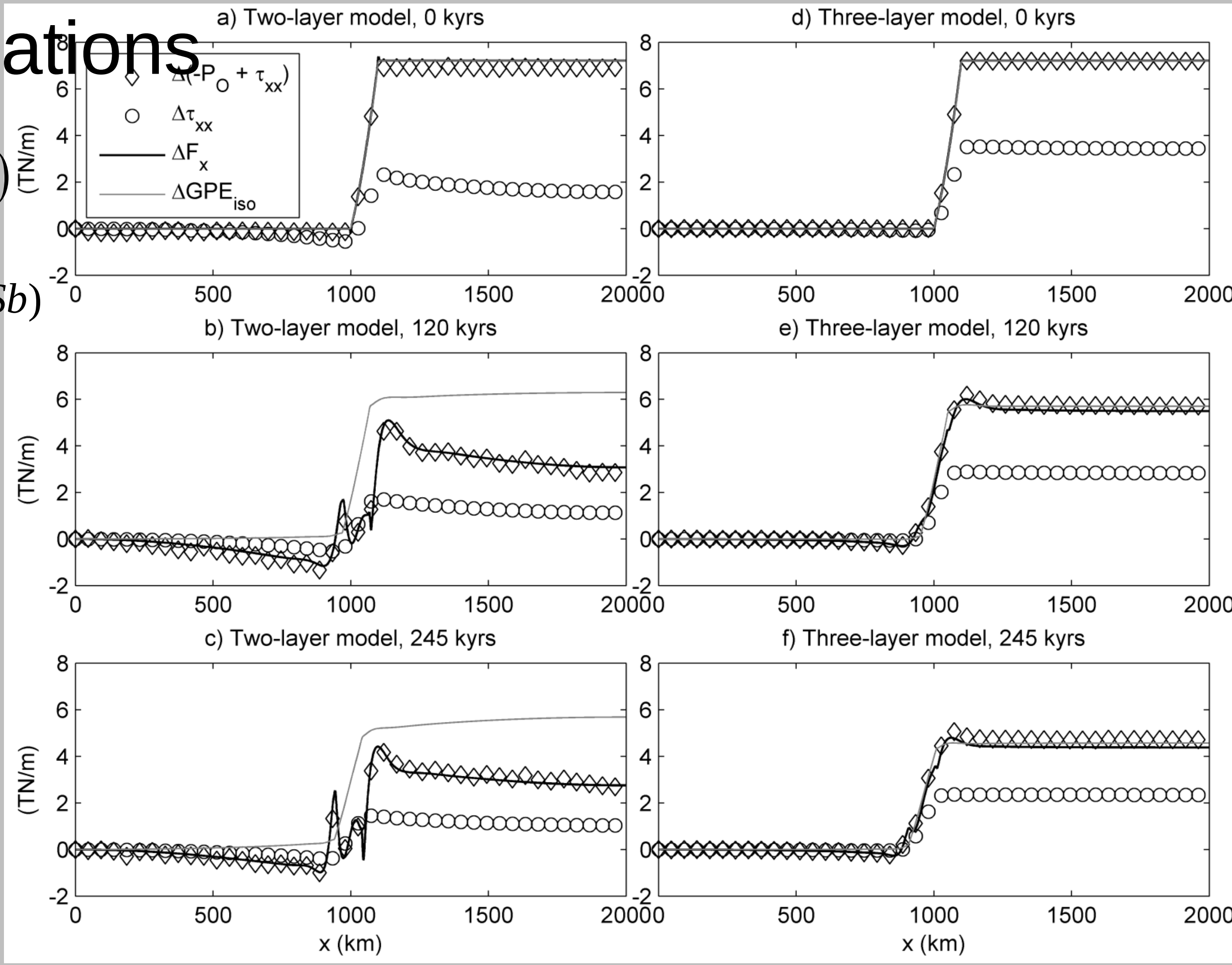


Top-to-base viscosity ratio: (a) 10; (b) 1000

Schmalholtz et al, 2014

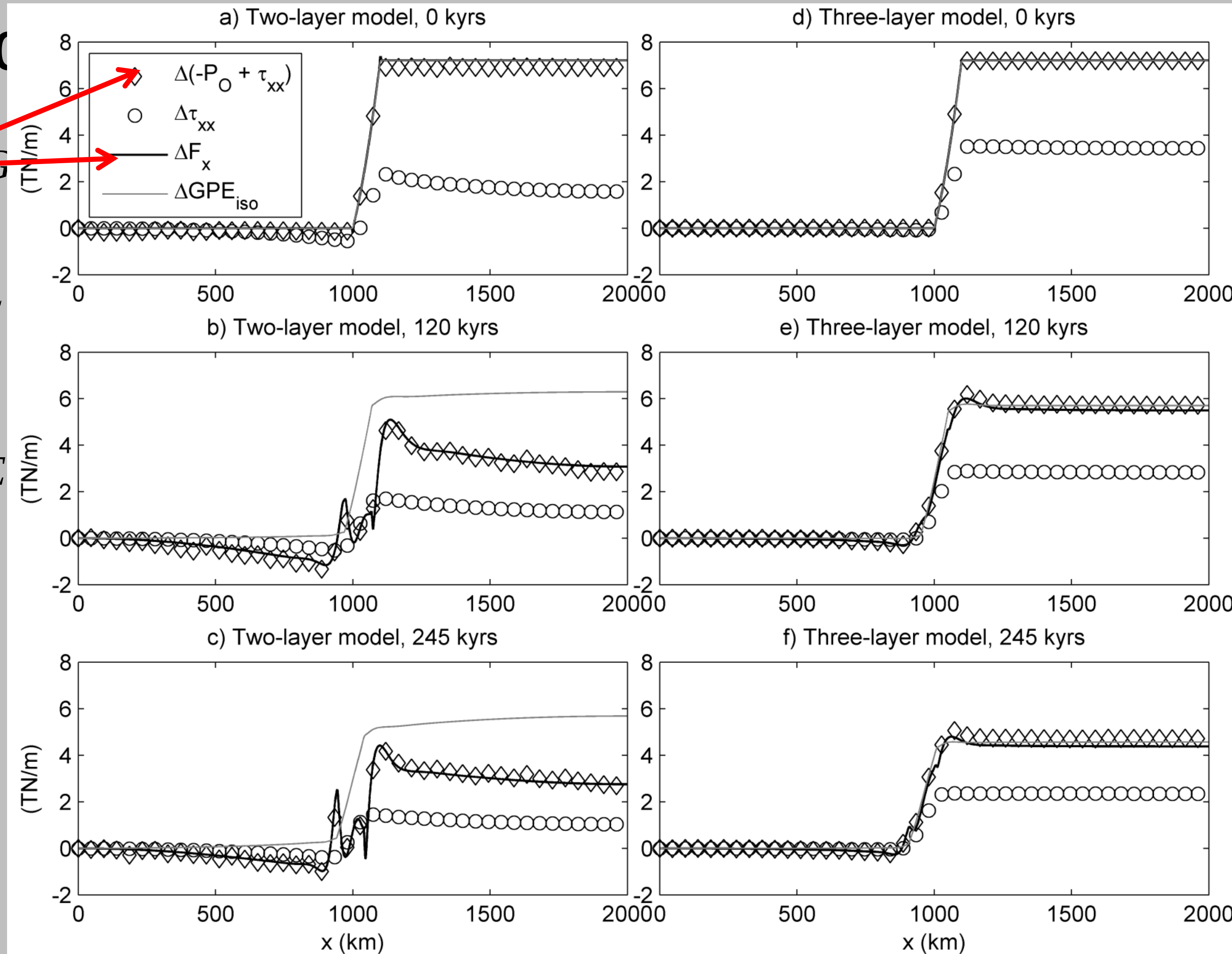
Testing ec

$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx} - \bar{Q}\right) = \frac{\partial}{\partial x}\left(G\right.$$

$$\frac{\partial}{\partial x}\left(\bar{\tau}_{xz}\right) = P(x,Sb) - P_L$$

$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx}\right) = \frac{\partial}{\partial x}\left(GPE\right.$$

$$P(x,Sb) = P_L(x,Sb)$$



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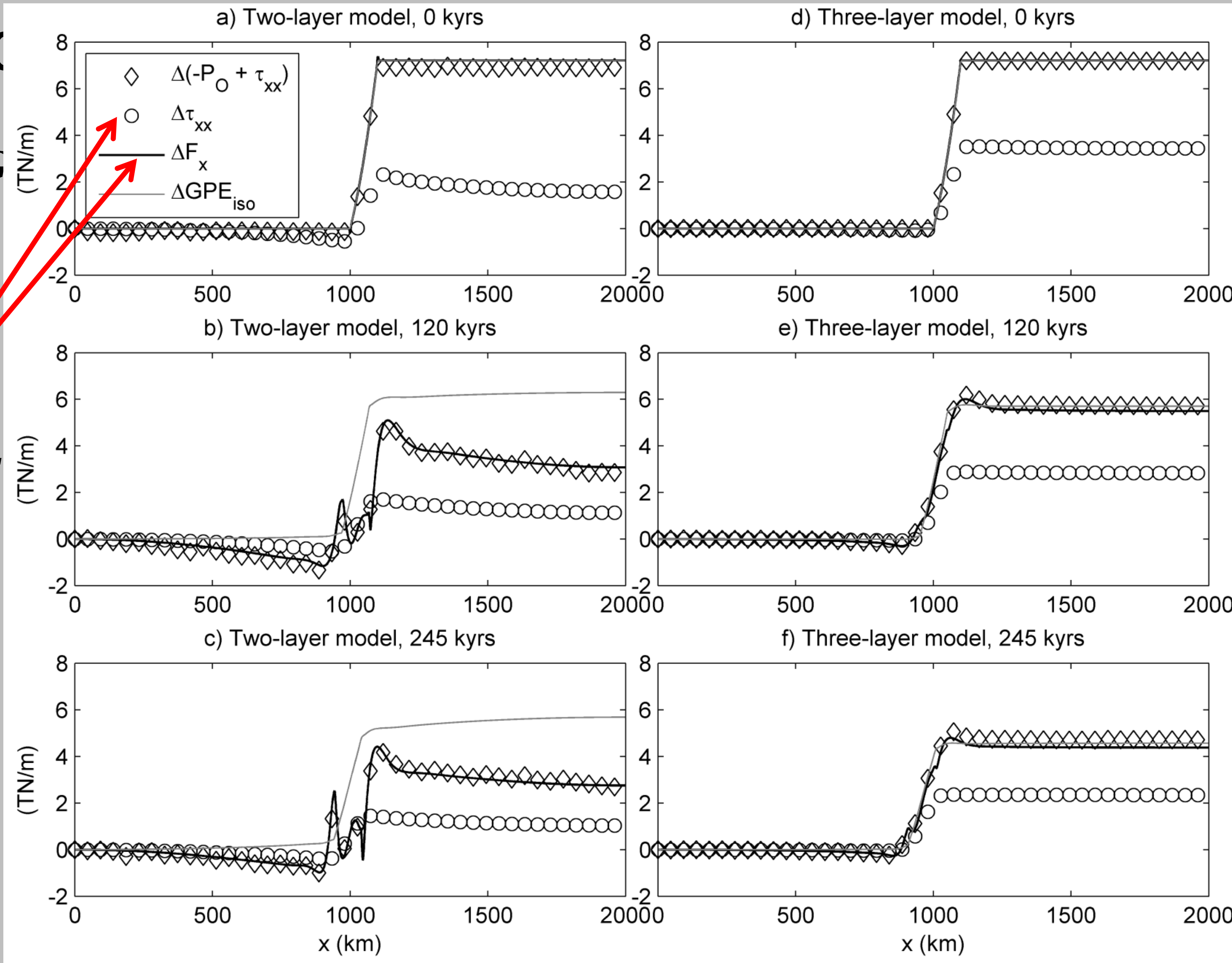

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Schmalholtz et al, 2014

Testing ec

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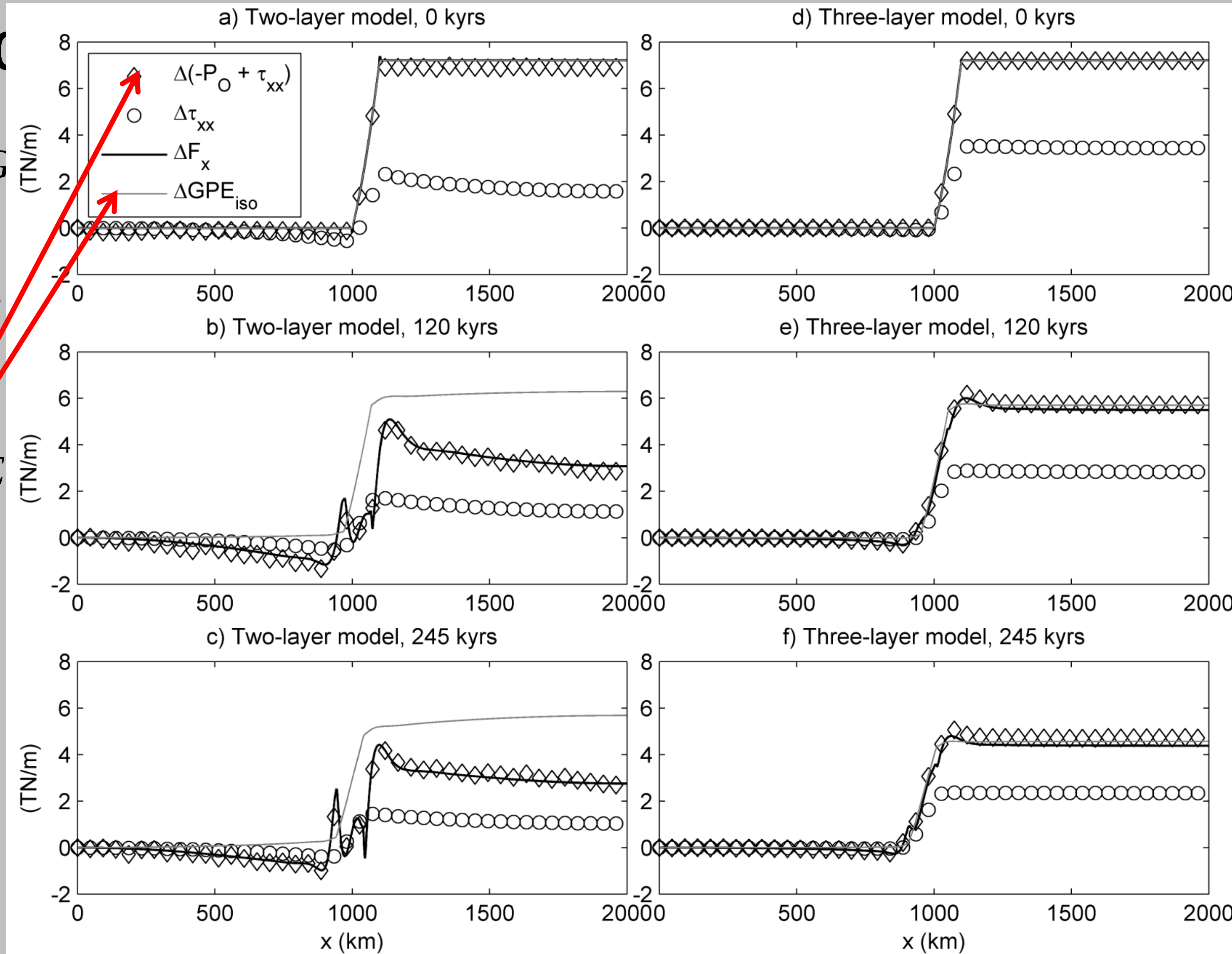

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Testing equati

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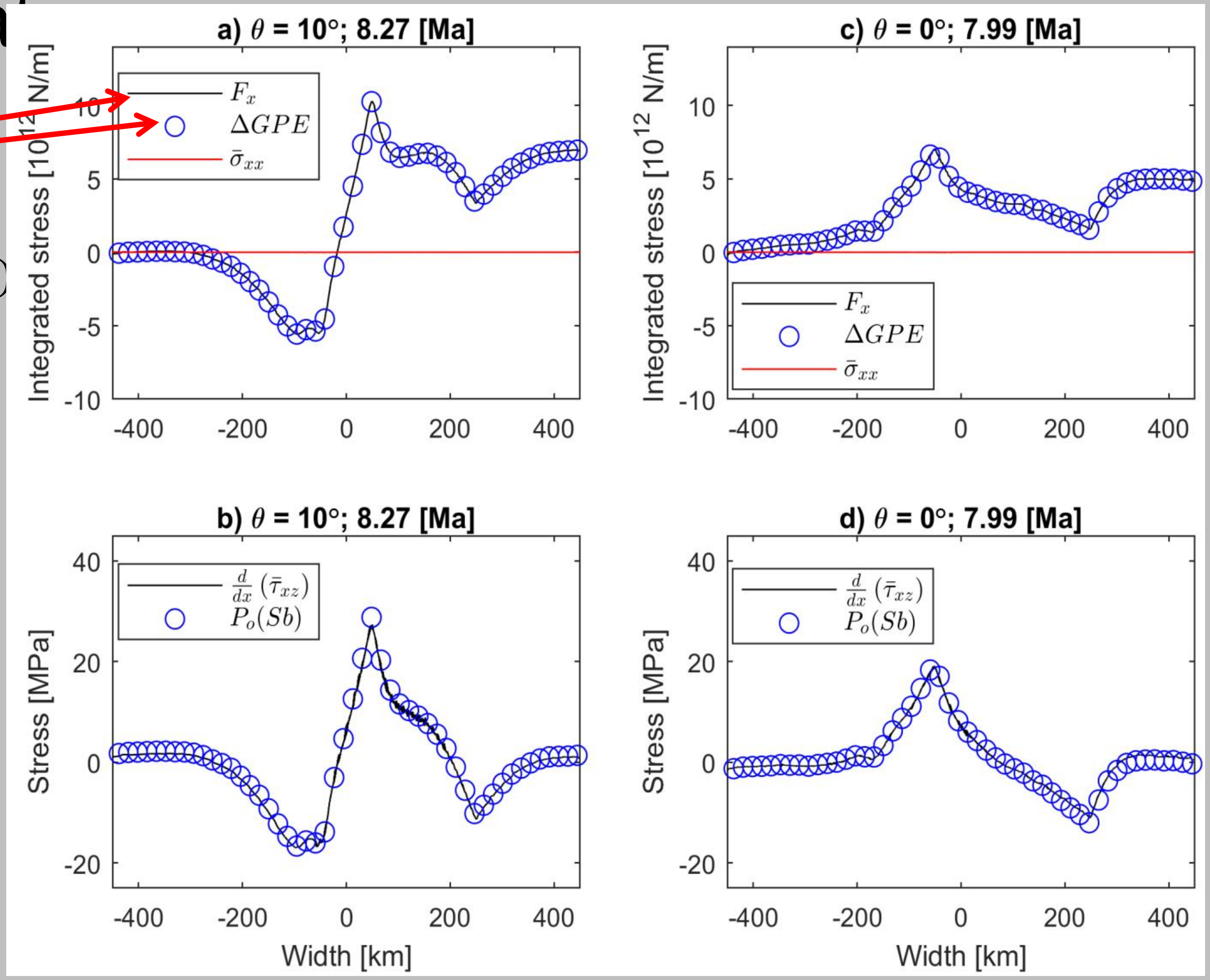

Visco-plastic rheology (different strength of crust)

Schmalholtz et al, in press

Testing equati

$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx}-\bar{Q}\right)=\frac{\partial}{\partial x}\left(GPE\right)$$

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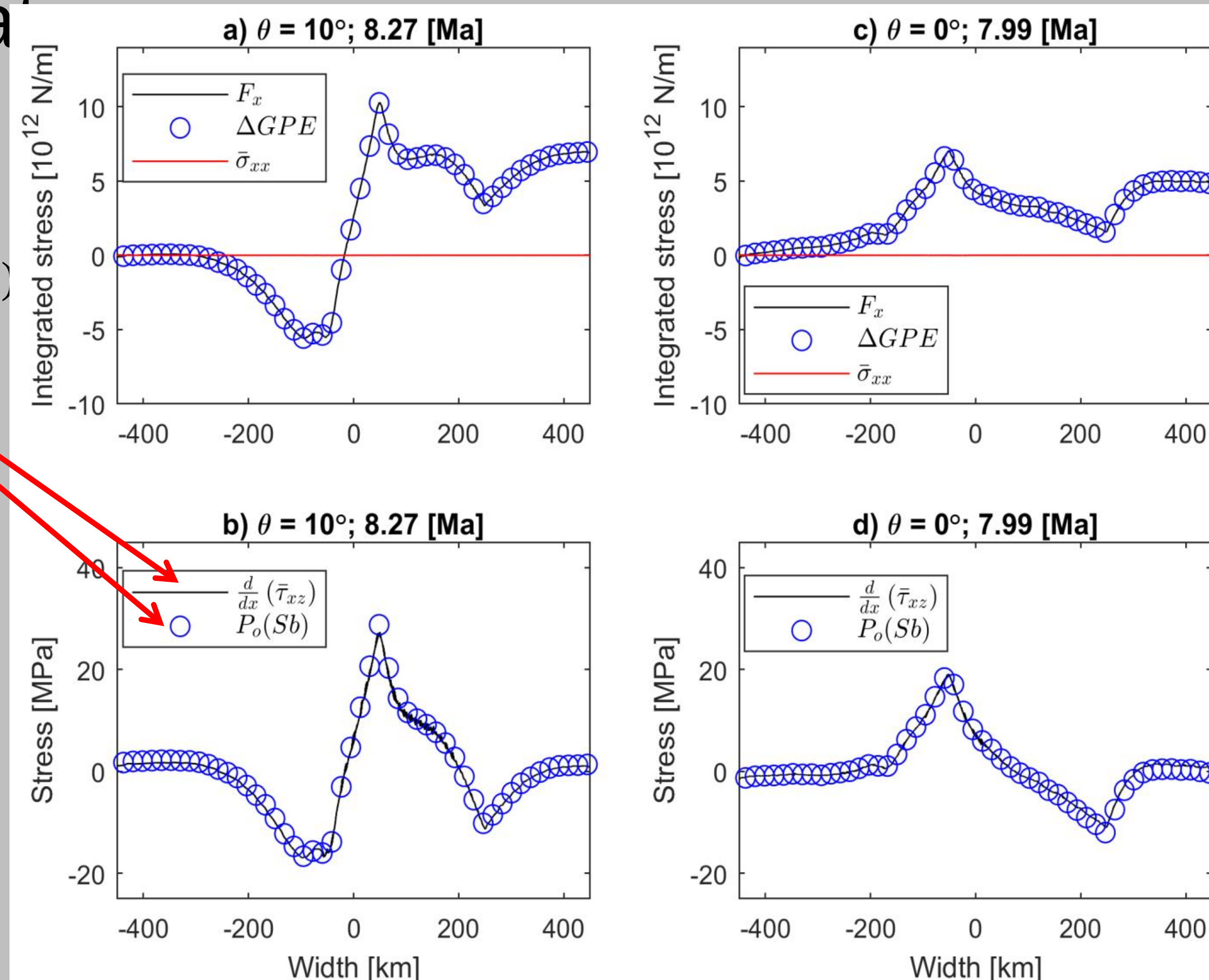

Visco-plastic rheology (different strength of crust)

Schmalholtz et al, in press

Thin sheet equations

$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx} - \bar{Q}\right) = \frac{\partial}{\partial x}\left(GPE\right)$$

$$\frac{\partial}{\partial x}\left(\bar{\tau}_{xz}\right) = P(x, Sb) - P_L(x, Sb)$$

$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx}\right) = \frac{\partial}{\partial x}\left(GPE\right)$$

$$P(x, Sb) = P_L(x, Sb)$$

- Thin sheet equations are correct while estimating results of fully-numerical calculations
- The equations are more precise for larger strength contrast (top/bottom)
- Gives us a tool for **rheology-independent estimations**
  - Even if asthenosphere would be inviscid, the local isostasy may be violated

The magnitude of the horizontal driving force per unit length, that is, the depth-integrated deviation of the horizontal total stress from the lithostatic pressure (or static stress), of approximately 7 TN m^{-1} resulting from the *GPE* variation related to the Tibetan Plateau is sufficient to fold the Indian oceanic lithosphere.

Schmalholtz et al, 2014

Characteristic stresses

$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx} - \bar{Q}\right) = \frac{\partial}{\partial x}\left(GPE\right)$$

$$\frac{\partial}{\partial x}\left(\bar{\tau}_{xz}\right) = P(x,Sb) - P_L(x,Sb)$$

$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx}\right) = \frac{\partial}{\partial x}\left(GPE\right)$$

$$P(x,Sb) = P_L(x,Sb)$$

Can these equations help us to estimate amplitude of stresses?

Schmalholtz et al, 2014

Characteristic stresses

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$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx}\right) = \frac{\partial}{\partial x}\left(GPE\right)$$

$$P(x,Sb) = P_L(x,Sb)$$

ACCEPTED MANUSCRIPT

Distribution and magnitude of stress due to lateral variation of gravitational potential energy between Indian lowland and Tibetan plateau

Stefan M Schmalholz, Thibault Duretz, György Hetényi, Sergei Medvedev

Geophysical Journal International, ggy463, https://doi.org/10.1093/gji/ggy463

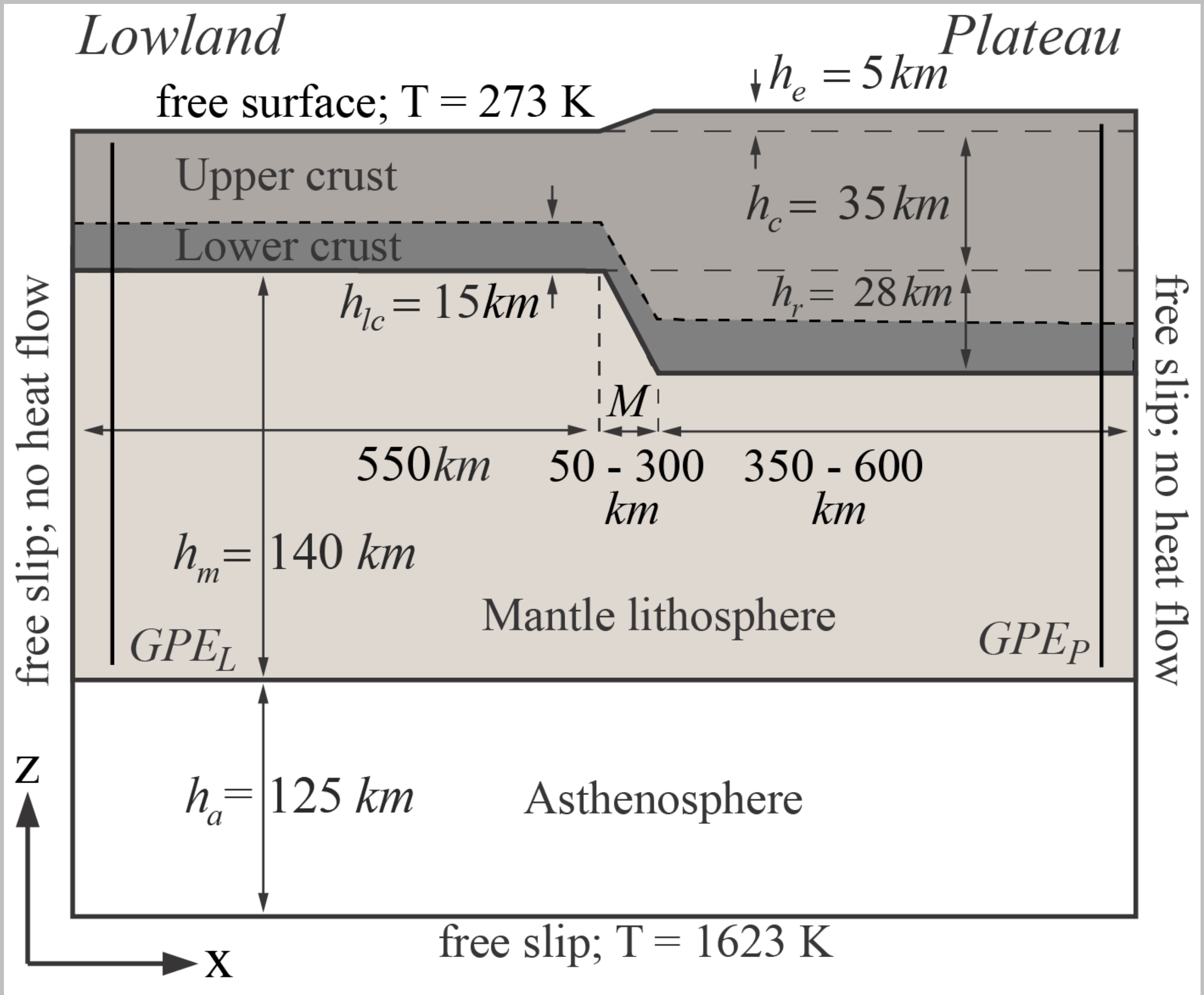
Published: 01 November 2018

Schmalholtz et al, in press

Characteristic stresses

Earthquake-based estimates (stress drop) ~10 MPa in the crust

- Small stresses?
- Weak crust?



Schmalholtz et al, in press

Characteristic stresses

Earthquake-based estimates (stress drop) ~10 MPa in the crust

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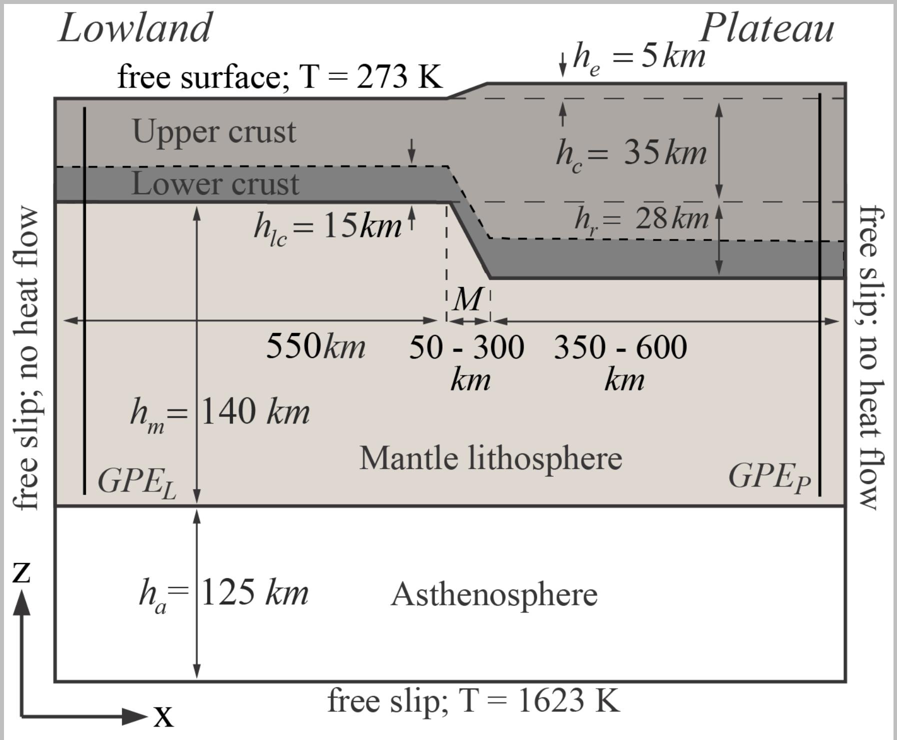
$$\frac{\partial}{\partial x}\left(2\overline{\tau}_{xx}\right)=\frac{\partial}{\partial x}\left(GPE\right)$$



Schmalholtz et al, in press

Characteristic stresses

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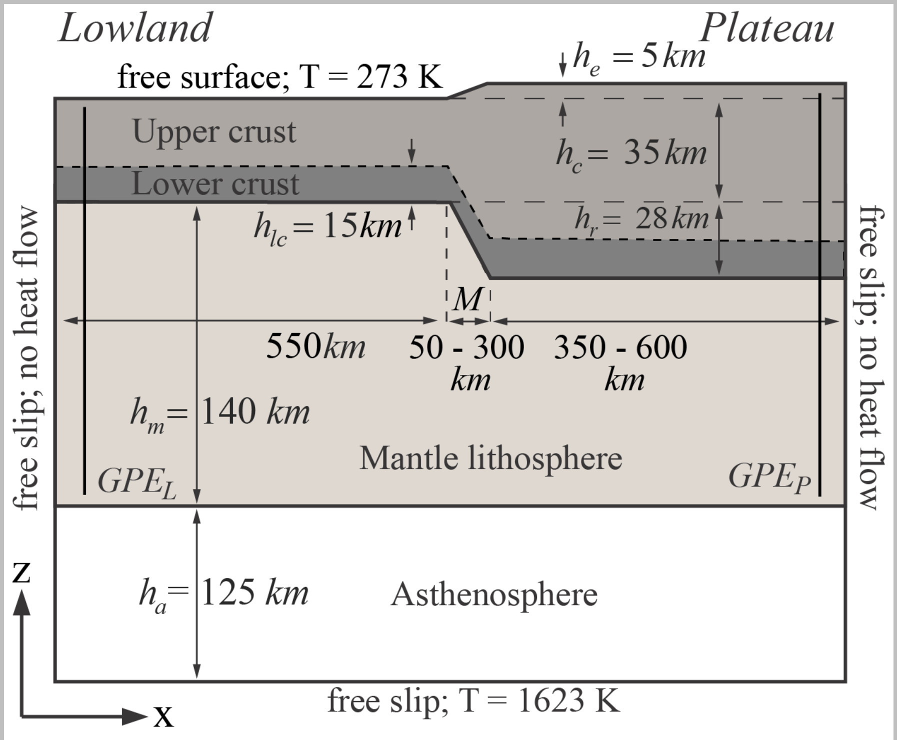


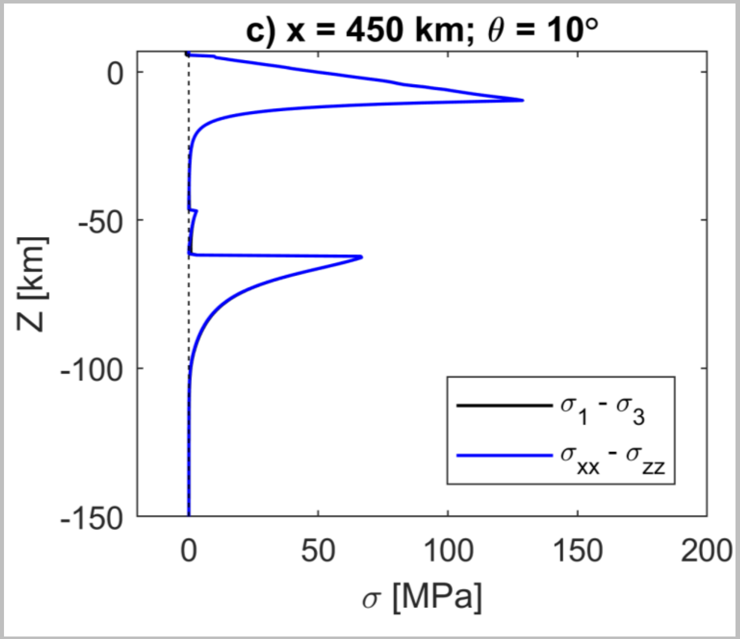
$$\frac{\partial}{\partial x}\left(2\overline{\tau}_{xx}\right) = \frac{\partial}{\partial x}\left(GPE\right) \longrightarrow 2\Delta\overline{\tau}_{xx} = \Delta GPE$$

$$\overline{\tau}_{xx} \approx 1700\ \text{MPa}\cdot\text{km}$$

Characteristic stresses

Earthquake-based estimates (stress drop) ~10 MPa in the crust

- Small stresses?
- Weak crust?



What is characteristic stress?

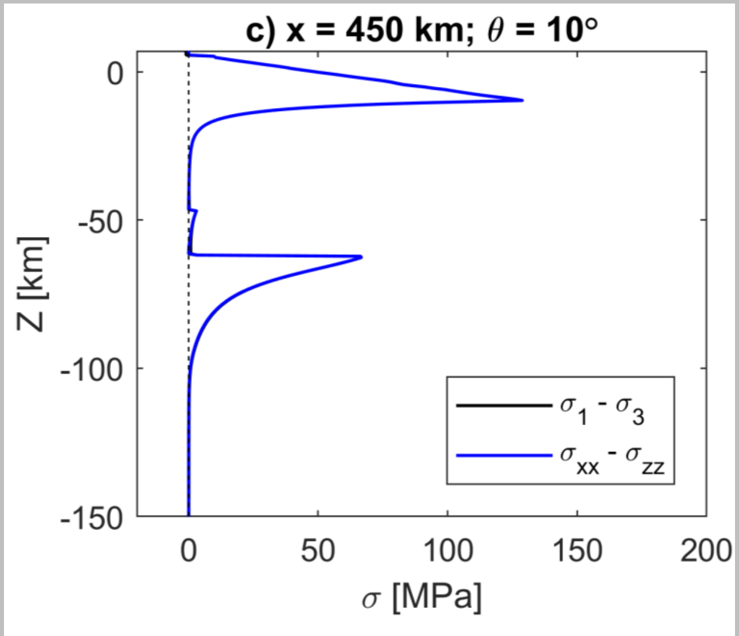
$$\frac{\partial}{\partial x}\left(2\bar{\tau}_{xx}\right) = \frac{\partial}{\partial x}\left(GPE\right) \longrightarrow 2\Delta\bar{\tau}_{xx} = \Delta GPE$$

$$\bar{\tau}_{xx} \approx 1700\ \mathrm{MPa \cdot km}$$

Characteristic stresses

Effective rheological thickness (ERT):

ERT= 35 – 45 km
Characteristic stress:
35 – 50 MPa



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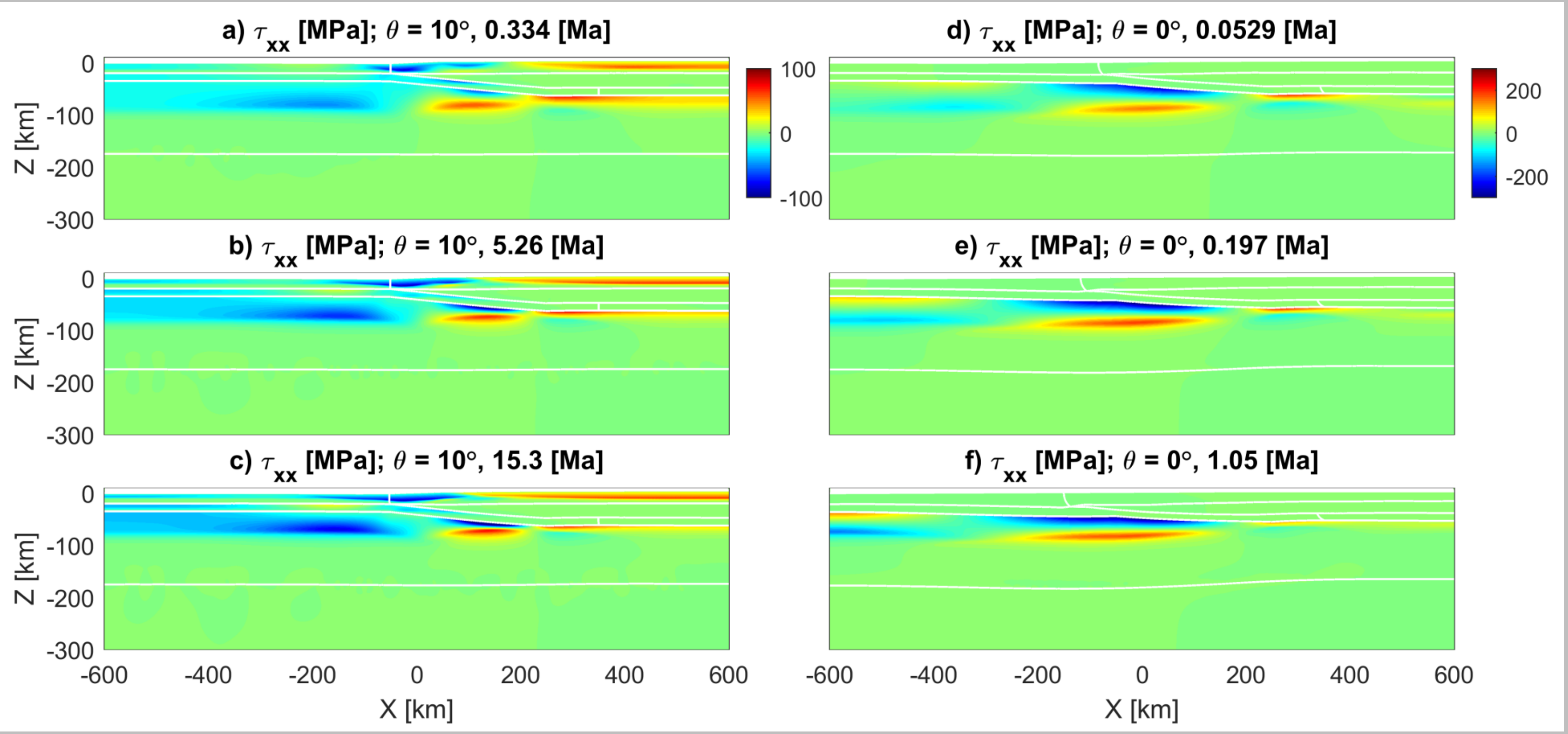
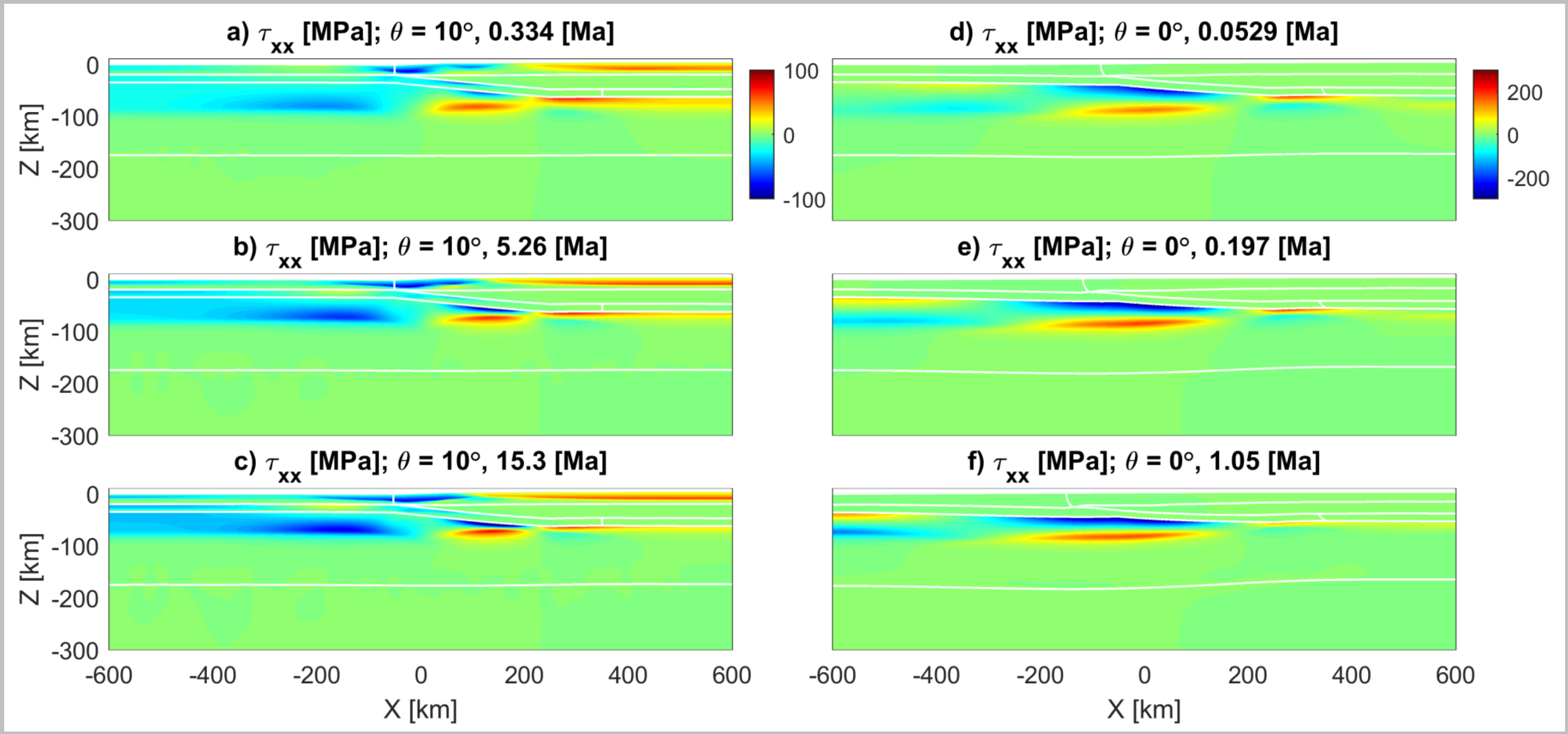
Characteristic stresses



Characteristic stress: 35 – 50 MPa

Characteristic stresses

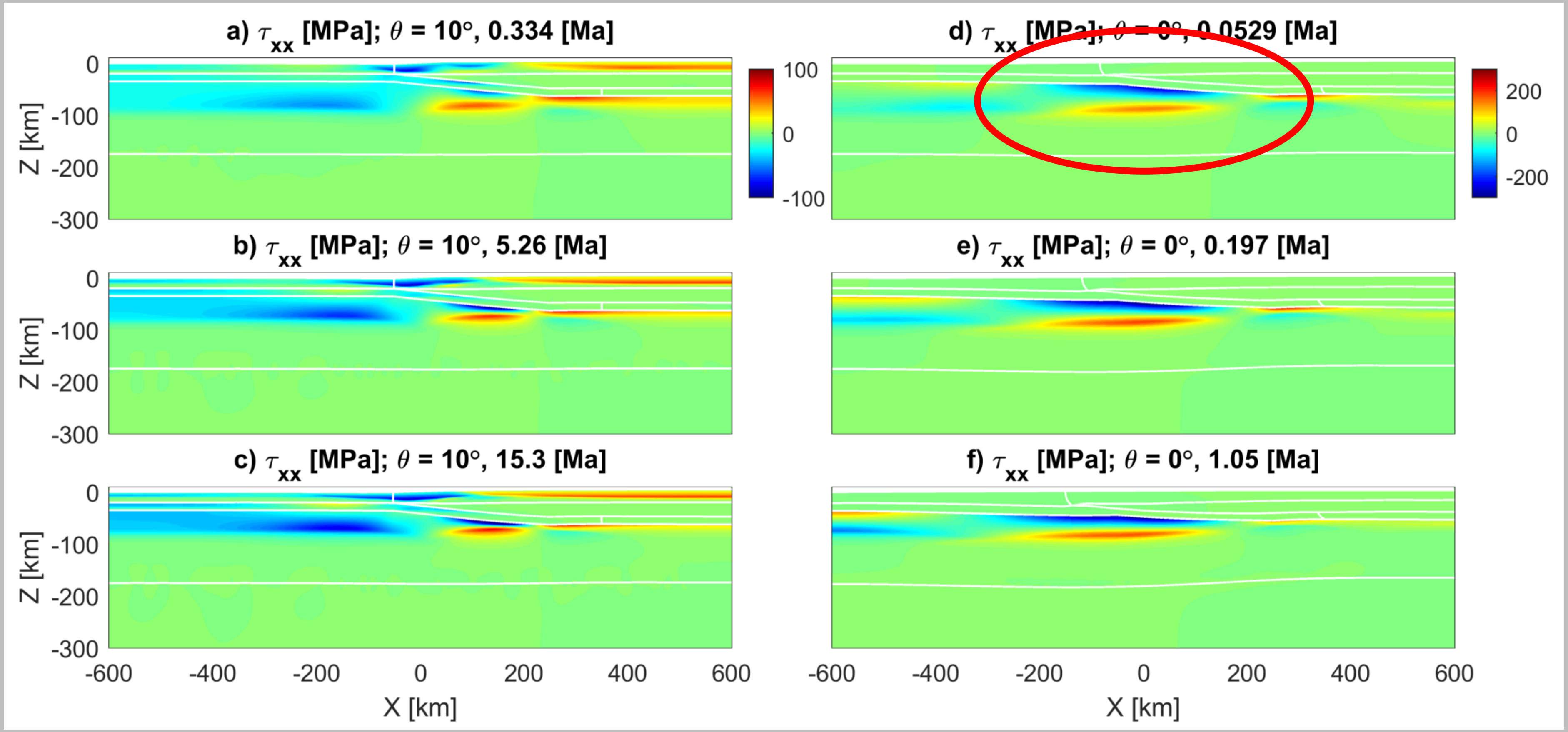


Strong crust

Weak crust

Characteristic stress: 35 – 50 MPa

Characteristic stresses

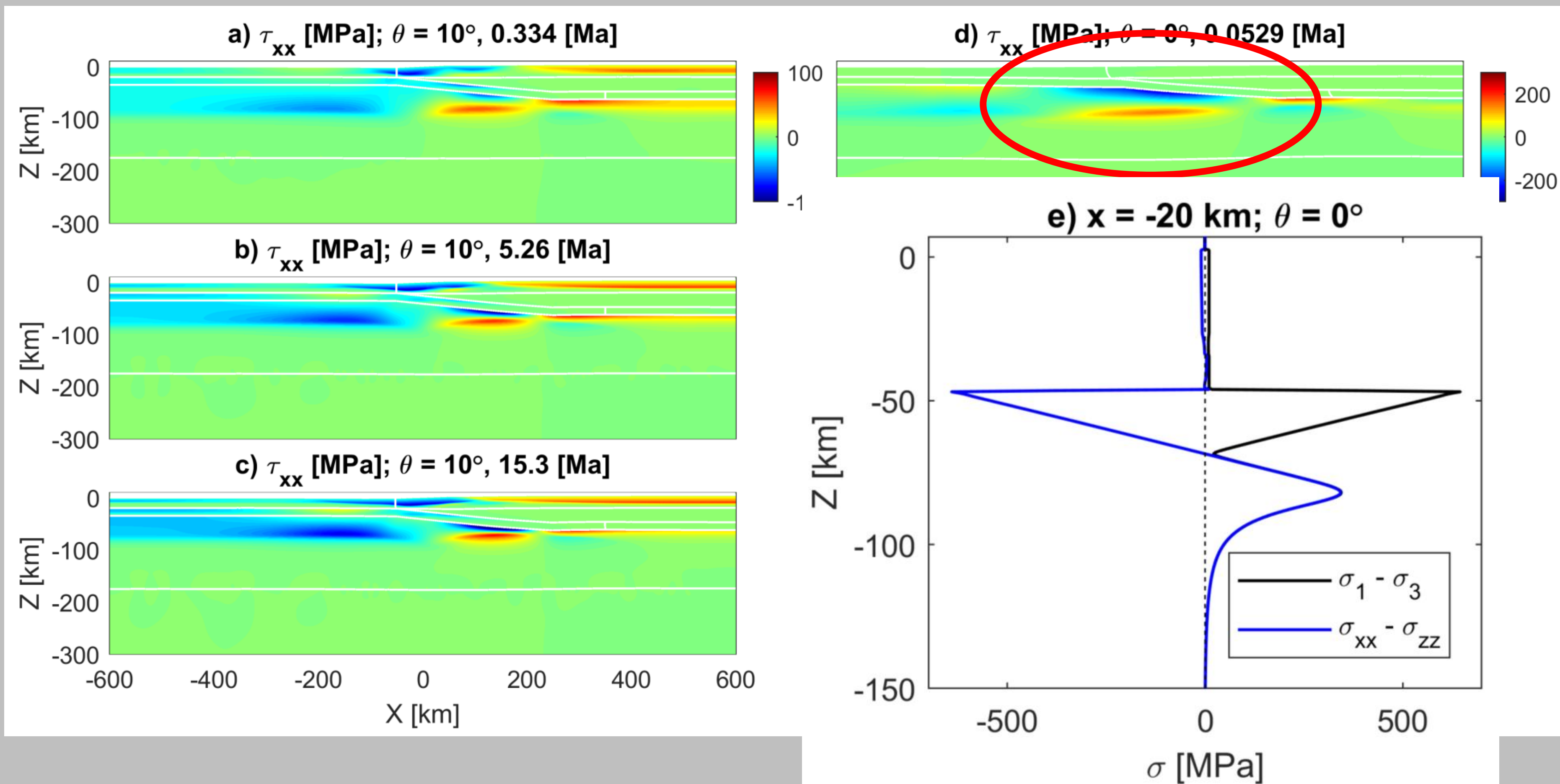


Strong crust

Characteristic stress: 35 – 50 MPa

Weak crust:
huge stresses in the subcrustal lithosphere

Characteristic stresses

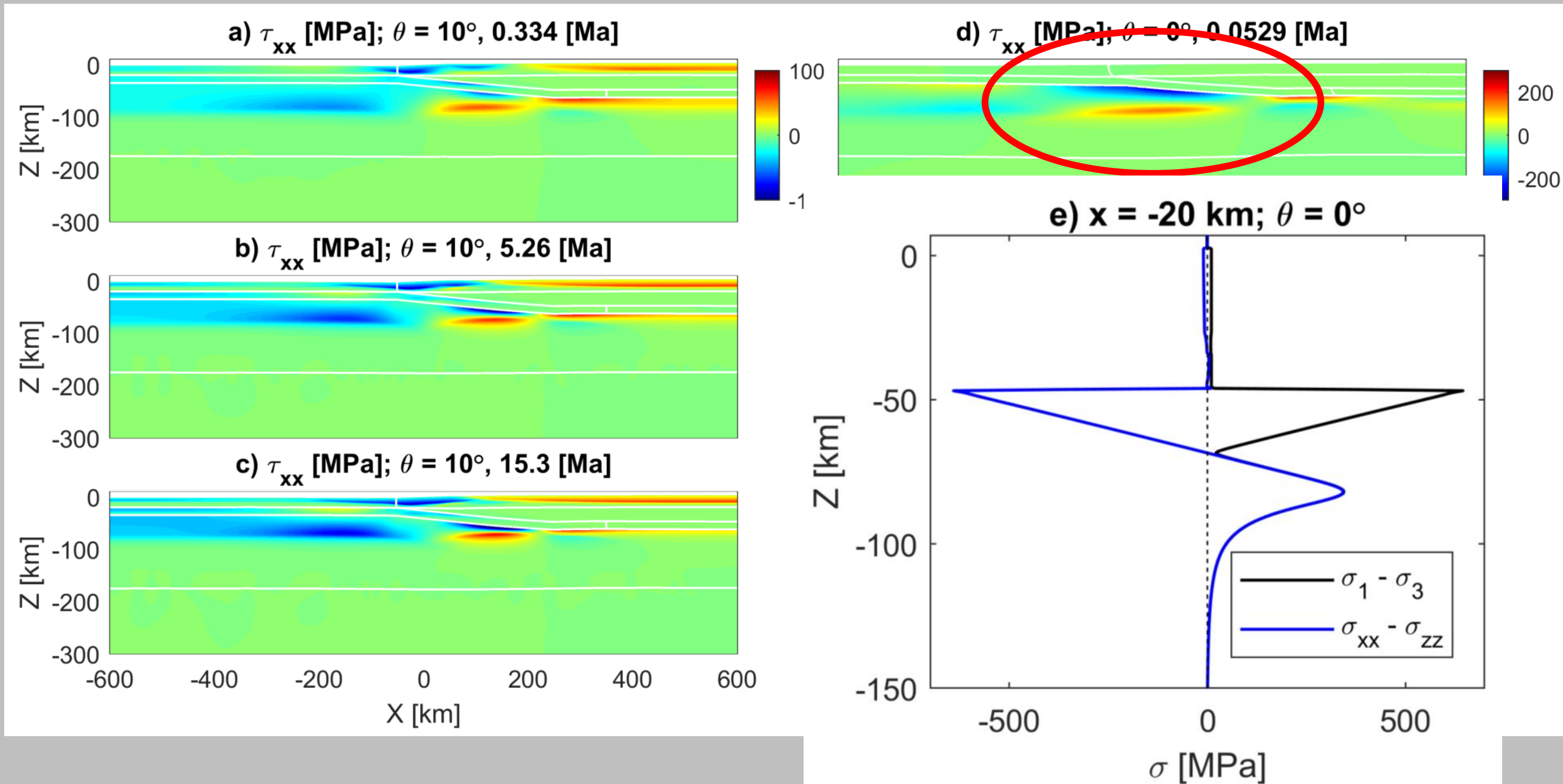


Strong crust

Characteristic stress: 35 – 50 MPa

Weak crust:
huge stresses in the subcrustal lithosphere

Bending stresses



Strong crust

Characteristic stress: 35 – 50 MPa

Weak crust:
huge stresses in the subcrustal lithosphere

Bending stresses

$$\frac{\partial}{\partial x}\left(\bar{\tau}_{xz}\right) = P(x, Sb) - P_L(x, Sb)$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

$$\overline{\tau_{iz}} = \int_{S_1}^{S_2} \tau_{iz}\, dz = (z_c \cdot \tau_{iz})|_{S_1}^{S_2} - \int_{S_1}^{S_2} z_c \cdot \frac{\partial \tau_{iz}}{\partial z}\, dz = (z_c \cdot \tau_{iz})|_{S_1}^{S_2} + \int_{S_1}^{S_2} z_c \cdot \left(-\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}\right) dz$$

$$= \overline{z_c T_j} + \left[\frac{\partial\, \overline{z_c \cdot \tau_{ij}}}{\partial x_j} - \frac{\partial \overline{z_c \cdot P}}{\partial x_i} + \overline{\tau_{ij}}\, \frac{\partial w}{\partial x_j} - \overline{P}\, \frac{\partial w}{\partial x_i}\right],$$

ETSA, 1999, eq. 12

Bending stresses

$$\frac{\partial}{\partial x}\left(\bar{\tau}_{xz}\right) = P(x,Sb) - P_L(x,Sb)$$

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ETSA, 1999, eq. 12

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Folding and necking across the scales: a review of theoretical and experimental results and their applications

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[2]Department of Earth Sciences, ETH Zurich, Zurich, Switzerland

Schmalholtz & Mantkelow, 2016

Bending stresses

$$\frac{\partial}{\partial x}\left(\bar{\tau}_{xz}\right) = P(x,Sb) - P_L(x,Sb)$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

$$\bar{\tau}_{xz} = \int_{Sb}^{St(x)} \tau_{xz} dz = \tau_{xz}\left(z-w\right)\Big|_{Sb}^{St(x)} - \int_{Sb}^{St(x)} \left(z-w\right)\frac{\partial \tau_{xz}}{\partial z} dz = \frac{\partial}{\partial x}\Pi\left(\sigma_{xx}\right) + \bar{\sigma}_{xx}\frac{\partial w}{\partial x}$$

$$\Pi\left(\sigma_{ij}\right) = \int_{Sb}^{St(x)} \sigma_{ij}\left(z-w\right)dz = \overline{\sigma_{ij}\left(z-w\right)}$$

Moment of stress around level *w(x)*

Bending stresses

$$\frac{\partial}{\partial x}\left(\bar{\tau}_{xz}\right) = P(x, Sb) - P_L(x, Sb)$$

$$\frac{\partial^2}{\partial x^2}\Pi\left(\sigma_{xx}\right) + \bar{\sigma}_{xx}\frac{\partial^2 w}{\partial x^2} = P(x, Sb) - P_L(x, Sb)$$

$$\frac{\partial^2}{\partial x^2}\Pi\left(\sigma_{xx}^d\right) - \frac{\partial^2}{\partial x^2}\Pi\left(P_L\right) + \bar{\sigma}_{xx}\frac{\partial^2 w}{\partial x^2} = P(x, Sb) - P_L(x, Sb)$$

$$\sigma_{xx} = \sigma_{xx}^d + \sigma_{xx}^{static} = \sigma_{xx}^d - P_L$$

Bending stresses

$$\frac{\partial^2}{\partial x^2}\Pi\left(\sigma_{xx}^{d}\right)-\frac{\partial^2}{\partial x^2}\Pi\left(P_L\right)+\bar{\sigma}_{xx}\frac{\partial^2 w}{\partial x^2}=P(x,Sb)-P_L(x,Sb)$$

No additional assumption!

Bending stresses

$$\frac{\partial^2}{\partial x^2}\Pi\left(\sigma_{xx}^d\right)-\frac{\partial^2}{\partial x^2}\Pi\left(P_L\right)+\bar{\sigma}_{xx}\frac{\partial^2 w}{\partial x^2}=P(x,Sb)-P_L(x,Sb)$$

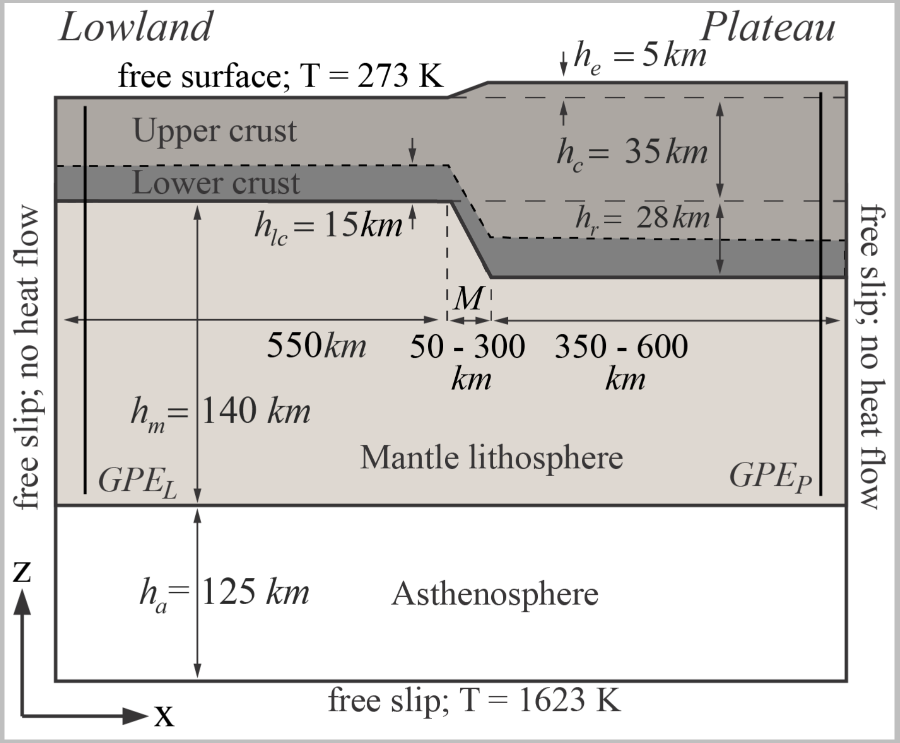
Now let's assume:
simple geometry at t=0,
Local isostasy

$$P(x,Sb)=P_L(x,Sb)$$

Piece-wise linear level

$$w=ax+b$$

$$\frac{\partial^2}{\partial x^2}\Pi\left(\sigma_{xx}^d\right)=\frac{\partial^2}{\partial x^2}\Pi\left(P_L\right)$$



Bending stresses

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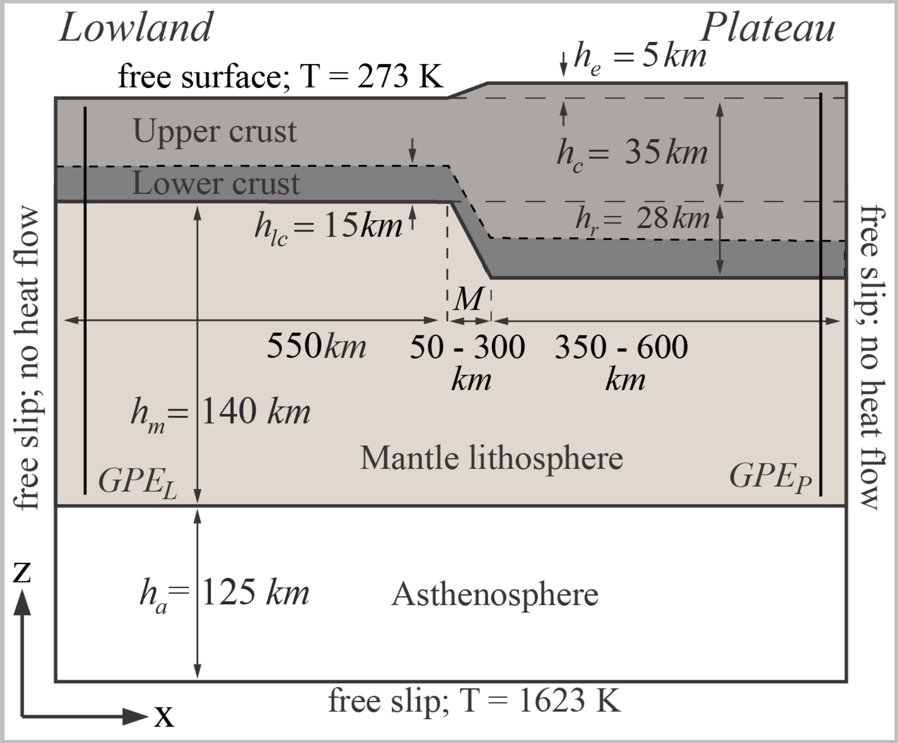


Moment of lithostatic pressure

$$\Pi(P_L) = \int_{Sb}^{St(x)} (z-w) \int_{z}^{St(x)} \rho(x,z') g dz' dz =$$

$$= \left(\frac{h_c(x)^3}{3} + \frac{h_m(x)^2 h_c(x)}{2} \right) \rho_c g + \frac{h_m(x)^3}{3} \rho_m g - \left[St(x) - w(x) \right] GPE$$

$$h_c(x) = h_c + h_{ex} \frac{\rho_m}{\rho_m - \rho_c}$$

$$h_m(x) = h_m - h_{ex} \frac{\rho_c}{\rho_m - \rho_c}$$

$$St(x) - w(x) = h_{ex} + W = h_{ex} + w_1 h_{ex} + w_0$$

$$\left(\frac{h_c(x)^3}{3} + \frac{h_m(x)^2 h_c(x)}{2} \right) \rho_c g + \frac{h_m(x)^3}{3} \rho_m g = h_{ex}^3 A_1 g + h_{ex}^2 B_1 g + h_{ex} C_1 + D_1$$

$$h_{ex} GPE = h_{ex}^3 A_2 g + h_{ex}^2 B_2 g + h_{ex} C_2$$

$$W \cdot GPE = h_{ex}^3 w_1 A_2 g + h_{ex}^2 g \left[w_1 B_2 + w_0 A_2 \right] + W \cdot C_2$$

$$\left[St(x) - w(x) \right] GPE = h_{ex}^3 (1 + w_1) A_2 g + h_{ex}^2 \left[(1 + w_1) B_2 + w_0 A_2 \right] g + \ldots$$

$$A_1 = \frac{\rho_c \rho_m}{(\rho_m - \rho_c)^3} \left(\frac{\rho_m^2}{3} + \frac{\rho_c \rho_m}{2} - \frac{\rho_c^2}{3} \right)$$

$$B_1 = \frac{h_c \rho_c}{(\rho_m - \rho_c)^2} \left(\rho_m^2 + \frac{\rho_c^2}{2} \right)$$

$$A_2 = \frac{\rho_c \rho_m}{2(\rho_m - \rho_c)}$$

$$B_2 = h_c \rho_c$$

$$\Pi(P_L) = h_{ex}^3 A + h_{ex}^2 B + \ldots$$

$$A = \left[A_1 - (1 + w_1) A_2 \right] g$$

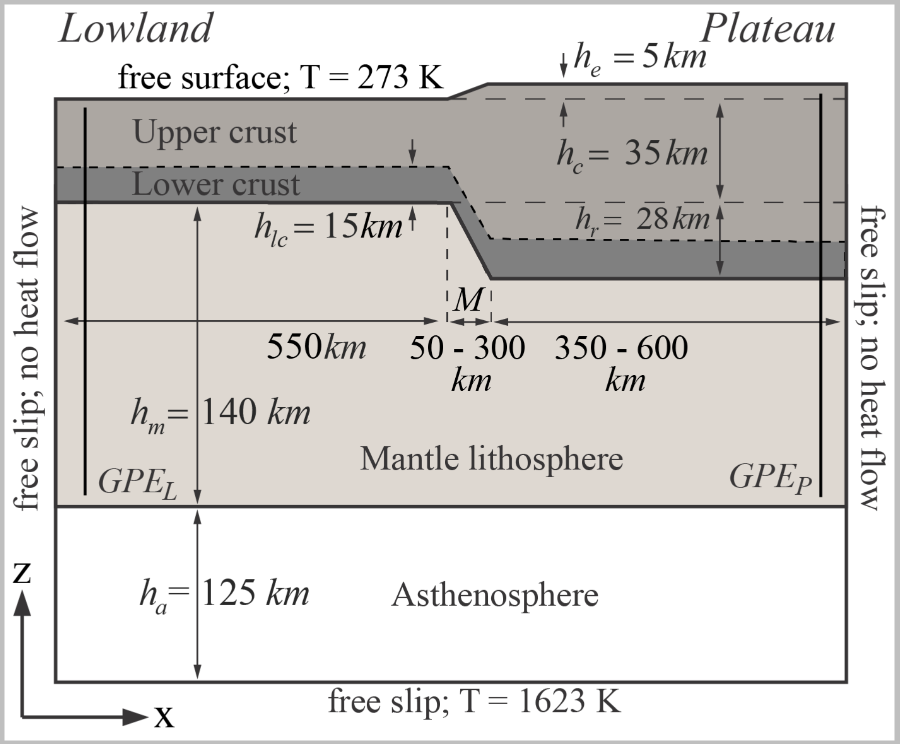
$$B = \left[B_1 - (1 + w_1) B_2 - w_0 A_2 \right] g$$

Bending stresses

$$\frac{\partial^2}{\partial x^2}\Pi\left(\sigma_{xx}^{d}\right)=\frac{\partial^2}{\partial x^2}\Pi\left(P_L\right)$$

$$\Pi\left(P_L\right)=h_{ex}^3 A+h_{ex}^2 B+\ldots$$

$$\Pi\left(\sigma_{xx}^{b}\right)=J\ h_{ex}\left[h_{ex}-h_e\right]\left[h_{ex}-K\right]$$



Bending stresses

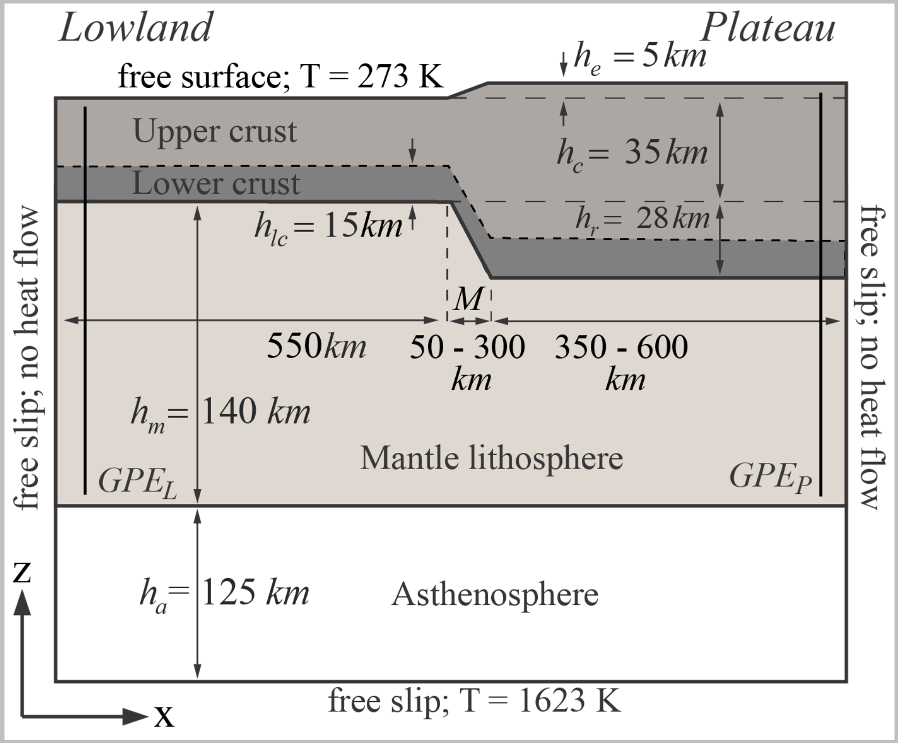
$$\frac{\partial^2}{\partial x^2}\Pi\left(\sigma_{xx}^d\right) = \frac{\partial^2}{\partial x^2}\Pi\left(P_L\right)$$

$$\Pi\left(P_L\right) = h_{ex}^3 A + h_{ex}^2 B + \ldots$$

$$\Pi\left(\sigma_{xx}^b\right) = J\ h_{ex}\left[h_{ex} - h_e\right]\left[h_{ex} - K\right]$$

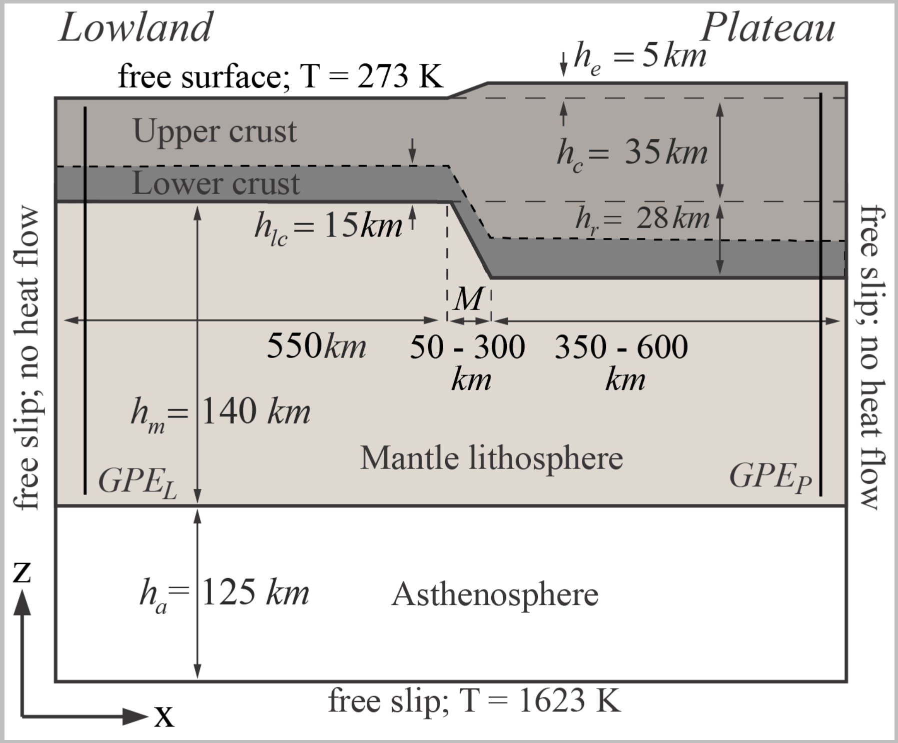
$$\sigma_{xx}^b \approx \pm\frac{6\Pi\left(\sigma_{xx}^b\right)}{ERT^2}$$

$$\tau_{xx}^b \approx \frac{\sigma_{xx}^b}{2} \approx \pm\frac{3\Pi\left(\sigma_{xx}^b\right)}{ERT^2}$$



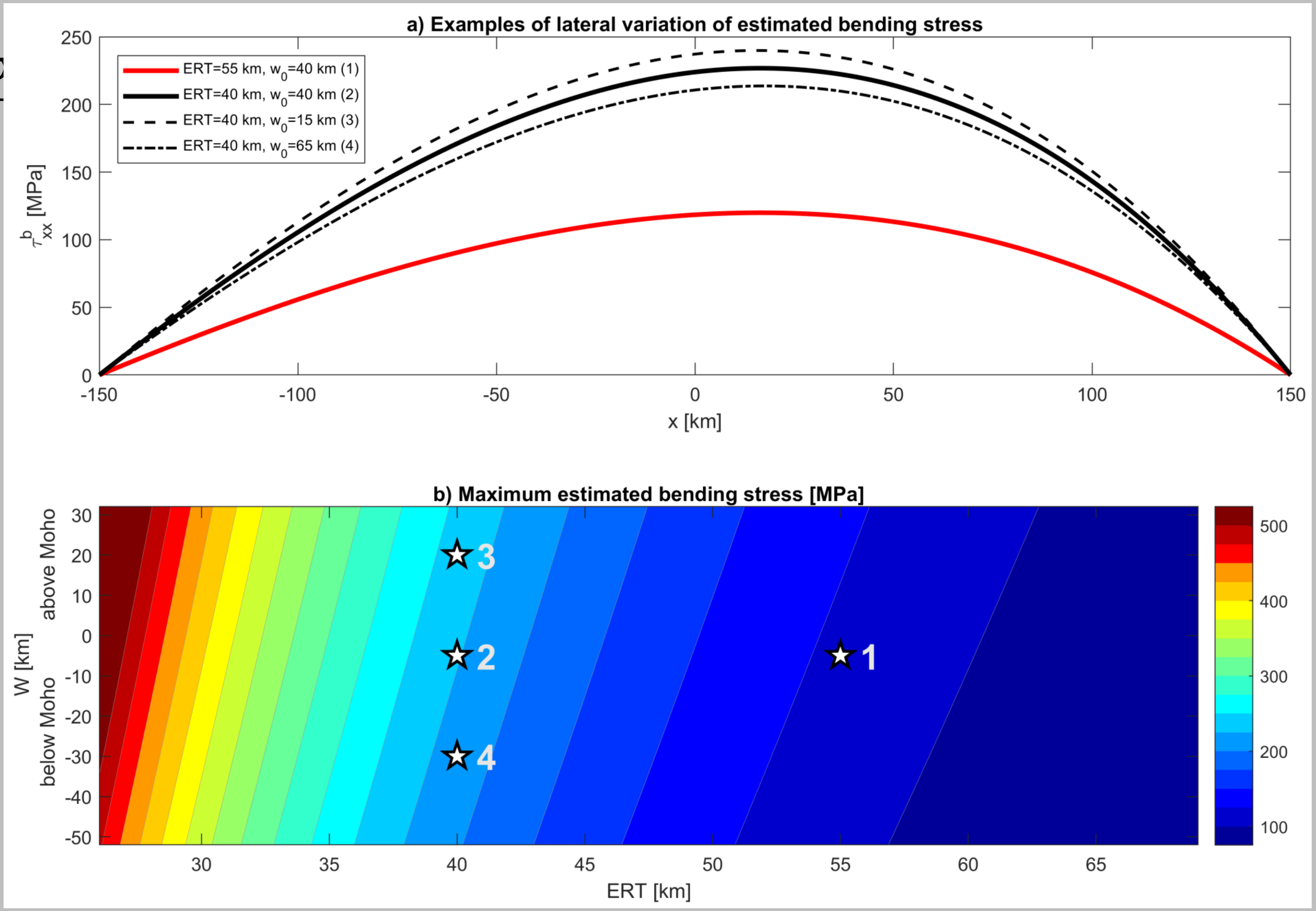
Bending stresses

$$\tau_{xx}^{b} \approx \frac{\sigma_{xx}^{b}}{2} \approx \pm\frac{3\Pi\left(\sigma_{xx}^{b}\right)}{ERT^{2}}$$

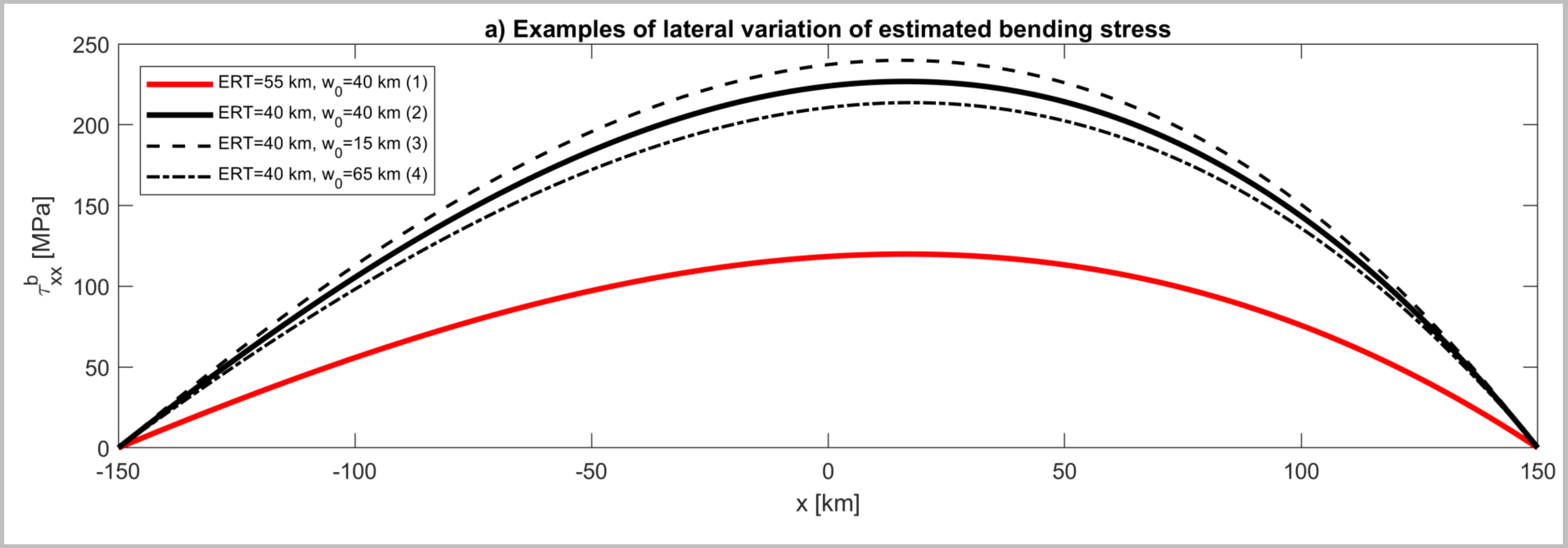


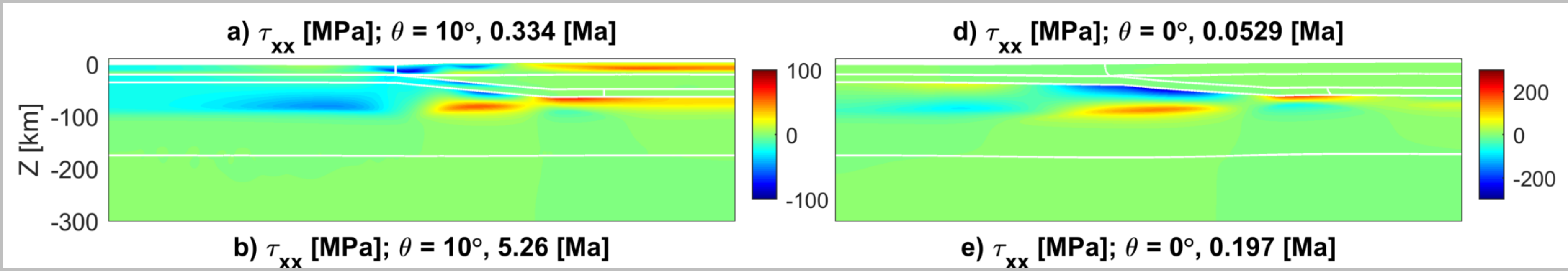
Bending stresses

$\tau_{xx}^{b} \approx$



Bending stresses





$$\tau_{xx}^{b} \approx \frac{\sigma_{xx}^{b}}{2} \approx \pm \frac{3\Pi\left(\sigma_{xx}^{b}\right)}{ERT^{2}}$$

Characteristic stresses

- Thin sheet approximation helps us to estimate characteristic stresses, even if they are dominated by bending moments
- The estimations presented are independent or weakly dependent on rheology
  - *ERT* is so far qualitative measure of stress-bearing layer thickness
  - *w* has rheologically loaded equation, but fortunately results are low dependent on *w*
- Thin sheet approximation is a useful tool and can be augmented or simplifyed